Walks obeying two-step rules on the square lattice: full, half and quarter planes

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Summary

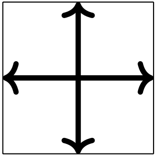
1. Introduction
   - Counting lattice paths
   - Lattice paths in the quarter plane
2. Two-step rules
   - Inspiration
3. Full plane
   - Direct enumeration
   - Generating functions
   - Asymptotics
   - Endpoint and drift
4. Upper half plane
   - Generating functions
   - Asymptotics
5. Quarter plane
   - Classifying
   - Functional equations
   - Groups
   - An orbit sum
   - Other cases
   - Computational results
6. Conclusion

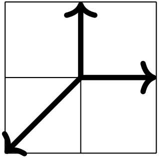
Introduction: Counting lattice paths

We are counting paths (not necessarily self-avoiding) on the square lattice $\mathbb{Z}^2$.

There are broadly two types of restrictions:

(1) the types of steps allowed
eg. on $\mathbb{Z}^2$, steps in $\mathcal{S} = \{(1,0),(0,1),(-1,0),(0,-1)\} = \{\text{E,N,W,S}\}$ or
$\mathcal{S} = \{(0,1),(1,0),(-1,-1)\} = \{\text{N,E,SW}\}$





(2) where the walks can go
eg. full plane, half-plane, quarter-plane, wedge of angle α

Counting may mean finding

(a) the number of m-step paths p_m starting at the origin, or

(b) the generating function $P(z) = \sum_{m \geq 0} p_m z^m$.

Often an expression for (a) leads to one for (b), but the reverse may be much harder.

May also want to restrict to paths which return to the origin, or end on a line, etc.

Introduction: Counting lattice paths

We can then ask questions like

- asymptotics of p_m as $m \to \infty$?
- properties of the generating functions? eg.
  - are they algebraic?
  - or differentially finite (D-finite) (ie. satisfy a linear ODE with polynomial coefficients)
  - or differentially algebraic (D-algebraic) (ie. $P(z)$ and its derivatives satisfy a polynomial)
- what does the "average walk" look like? where is the endpoint?
- efficient random sampling?

Many many papers and books over the last $\sim$20 years: I. Gessel, P. Flajolet, M. Bousquet-Mélou, M. Mishna, A. Rechnitzer, K. Raschel, A. Bostan, I. Kurkova, S. Melczer, C. Banderier, M. Singer, C. Hardouin, M. Albert, M. Kauers, O. Bernardi, ...

Methods use analytic and bijective combinatorics, probability theory, computer algebra, differential equations, Galois theory, elliptic curves, ...

Lattice paths in the quarter plane

There has been particular interest in walks with small step sets, ie. steps from a subset of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, restricted to a quarter plane.

Theorem (Bousquet-Mélou & Mishna 2010)

There are 79 non-isomorphic 'interesting' step sets for walks in the quarter plane.

To each step set $\mathcal{S}$ we associate the step generator

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j$$

eg. $\mathcal{S} = \{(1, 0), (0, 1), (-1, -1)\} \Rightarrow S(x, y) = x + y + \overline{xy}$.

Then there is a pair of involutions

$$(x, y) \mapsto (\sigma_x, y) \qquad \text{and} \qquad (x, y) \mapsto (x, \sigma_y)$$

which preserve $S(x, y)$. Together they generate a group $\mathcal{G}$ which is isomorphic to a (possibly infinite) dihedral group.

eg. For the step set above the involutions are $(x, y) \mapsto (\overline{xy}, y)$ and $(x, y) \mapsto (x, \overline{xy})$, generating the group (isomorphic to D_3)

$$G = \{(x, y), (\overline{xy}, y), (y, \overline{xy}), (y, x), (\overline{xy}, x), (x, \overline{xy})\}$$

Leads to

Theorem (various)

Of the 79 models in the quarter plane, 23 have finite groups and 56 have infinite groups. The generating functions of the 23 with finite groups are D-finite (four are algebraic), while the generating functions for the 56 with infinite groups are non-D-finite.

Functional equations & orbit sums

Let $Q(t;x,y) \equiv Q(x,y)$ be the generating function for walks in the quarter plane with t tracking length and x, y tracking endpoint coordinate. Then

Theorem

$$Q(x,y) = 1 + tS(x,y)Q(x,y) - t\overline{y}A_-(x)Q(x,0) - t\overline{x}B_-(y)Q(0,y) + t\overline{xy}C_-Q(0,0).$$

where

- $A_-(x) = [y^{-1}]S(x,y)$
- $B_-(y) = [x^{-1}]S(x,y)$
- $C_- = [x^{-1}y^{-1}]S(x,y)$

Typically written as

$$K(x,y)Q(x,y) = 1 - t\overline{y}A_-(x)Q(x,0) - t\overline{x}B_-(y)Q(0,y) + t\overline{xy}C_-Q(0,0). \quad (1)$$

where $K(x,y) \equiv K(t;x,y) = 1 - tS(x,y)$.

For 19 of the 23 models with finite groups, the full orbit sum method can be used to solve $Q(x, y)$.

Theorem (Bousquet-Mélou & Mishna 2010)

Define

$$R(t; x, y) = \frac{1}{K(x, y)} \sum_{g \in \mathcal{G}} \operatorname{sgn}(g) g(xy)$$

Then for all models with finite groups except those with $S = \{x, y, \overline{xy}\}$, $\{\overline{x}, \overline{y}, xy\}$, $\{x, y, \overline{x}, \overline{y}, xy, \overline{xy}\}$ *and* $\{x, \overline{x}, xy, \overline{xy}\}$,

$$Q(t; x, y) = \overline{xy}[x^{>} y^{>}]R(t; x, y).$$

Obtained by applying each group element to the functional equation, taking a linear combination which eliminates all $Q(*, 0)$ and $Q(0, *)$ terms, and then taking the $[x^{>} y^{>}]$ part which eliminates all other unknowns except $Q(x, y)$.

The following is useful:

Proposition (Bousquet-Mélou & Mishna 2010)

Let $F(t; x, y)$ be a series in t with coefficients in $\mathbb{C}(x)[y, \overline{y}]$. If $[y^{>}]F(t; x, y)$ has coefficients in $\mathbb{C}[x, \overline{x}, y]$, then $[x^{>}y^{>}]F(t; x, y)$ is D-finite.

These conditions can easily be checked for $R(t; x, y)$ above, and so it follows that for the 19 aforementioned models, $Q(t; x, y)$ is D-finite.

Algebraic models

The four remaining models all turn out to be algebraic.

For three of them $S = \{x, y, \overline{xy}\}$, $\{\overline{x}, \overline{y}, xy\}$ and $\{x, y, \overline{x}, \overline{y}, xy, \overline{xy}\}$, a slightly more complicated method called the half orbit sum can be used. Basic idea (Bousquet-Mélou & Mishna 2010):

- Apply three of the six group elements to the functional equation, and take a linear combination to cancel most of the unknowns on the RHS.
- Extract the $[y^0]$ part of this – a factor of $1/\sqrt{\Delta(x)}$ naturally emerges, where $\Delta(x)$ is the discriminant of $K(x, y)$.
- Factorise $\Delta(x)$ (the canonical factorisation of Gessel (1980)).
- Separate the $[x^>]$ and $[x^<]$ parts, obtaining $Q(x, 0)$.
- Use symmetries and back-substitution to get $Q(x, y)$.

The last model $S = \{x, \overline{x}, xy, \overline{xy}\}$ (Gessel's walks) is much more complicated, and has a storied history which will not be discussed here.

Two-step rules

Definition

A two-step rule $\mathcal{R}$ is a mapping

$$\mathcal{R} : \{\text{east, north, west, south}\}^2 \mapsto \{0, 1\},$$

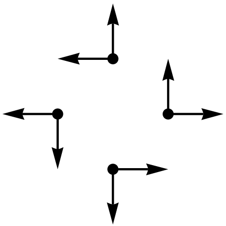
where $\mathcal{R}(i,j) = 1$ if step j can follow step i, and $\mathcal{R}(i,j) = 0$ if not.

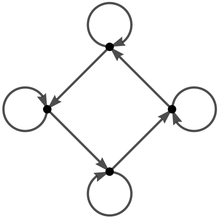
Let $\mathcal{T}$ be the set of all two-step rules.

$\mathcal{R}$ can be represented with a transfer matrix

$$\mathbf{T} \equiv \mathbf{T}(\mathcal{R}) = [\mathcal{R}(i,j)]_{\text{E,N,W,S}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

or diagramatically

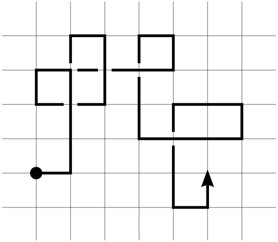


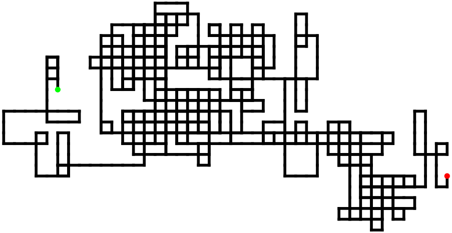


Definition

A walk $w = (v_1, v_2, \dots, v_m) \in \{E, N, W, S\}^*$ on the edges of the square lattice obeys $\mathcal{R}$ if $\mathcal{R}(v_i, v_{i+1}) = 1$ for $i = 1 \dots m - 1$.

Let $\mathcal{W} \equiv \mathcal{W}(\mathcal{R})$ be the set of walks obeying $\mathcal{R}$, and let $\mathcal{W}_m \subset \mathcal{W}$ be the walks of length m.





Define $a_m \equiv a_m(\mathcal{R})$ to be the number of walks (up to translation) of length m obeying $\mathcal{R}$.

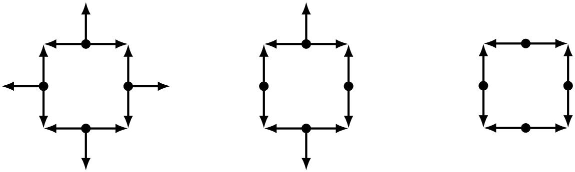
Then we can ask the same kinds of questions for these sequences.

Inspiration

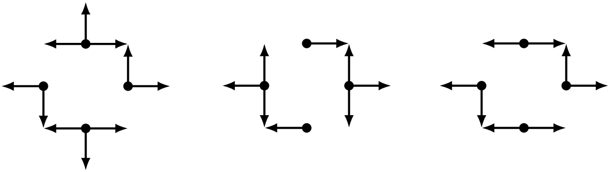
Guttmann, Prellberg and Owczarek (1993): considered self-avoiding walks obeying two-step rules.

11 non-isomorphic cases (with additional restrictions), with 4 universality classes based on length-scale exponents.

eg. These all (conjecturally) have a single length scale exponent $\nu = \frac{3}{4}$:



while these have a pair of exponents $\nu_{||} \approx 0.845$ and $\nu_{\perp} \approx 0.423$:



Classification & isomorphisms

$|\mathcal{T}| = 2^{16} = 65536$, but many rules are trivial.

Definition

A two-step rule $\mathcal{R}$ is connected, if for $i, j \in \{\mathrm{E}, \mathrm{N}, \mathrm{W}, \mathrm{S}\}$, there is a walk of length ≥ 1 obeying $\mathcal{R}$ which starts with i and ends with j.

Let $\mathcal{C}$ be the set of connected rules.

Equivalently, for each pair i, j there exists a $k \geq 1$ such that $(\mathbf{T}^k)_{ij} \geq 1$; or, the digraph must be strongly connected.

For simplicity we also rule out any implicit periodicity.

Definition

A connected two-step rule is aperiodic if there exists a $k \geq 1$ such that for all pairs i, j, $(\mathbf{T}^k)_{ij} \geq 1$.

Let $\mathcal{A}$ be the set of aperiodic rules.

Equivalently, $\mathbf{T}$ is primitive.

Proposition

Of the 65536 connected two-step rules, 25575 are aperiodic.

Lots of redundancy here.

Definition

Two rules $\mathcal{R}$ and $\mathcal{R}'$ with walk sets $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic if there exists a permutation π of $\{\mathrm{E}, \mathrm{N}, \mathrm{W}, \mathrm{S}\}$ with $\pi(\mathcal{W}) = \mathcal{W}'$.

In the full plane, this is the same as saying that we can permute the rows and columns of **T** to get **T**$'$; or that the digraphs are isomorphic. Not the case in the half- or quarter-plane!

Via Burnside's lemma:

Proposition

In the full plane, there are 1159 non-isomorphic aperiodic two-step rules.

Enumeration in the full plane

Let e_m be the number of walks of length m ending with an east step, and likewise n_m, w_m, s_m. Take $c_m = (e_m, n_m, w_m, s_m)$ and $p_m = e_m + n_m + w_m + s_m$.

Then set $c_1 = (1, 1, 1, 1)$ (ie. walks can start in any direction), and

$$c_m = c_{m-1} \cdot \mathbf{T} \qquad \text{for } m \geq 2.$$

By induction,

$$c_m = c_1 \cdot \mathbf{T}^{m-1} \qquad \text{for } m \geq 1.$$

Generating functions in the full plane

Define the generating functions

$$F_\theta(x,y) \equiv F_\theta(t;x,y) = \sum_m \hat{\theta}_m(x,y)t^m = \sum_{m,a,b} \theta_m(a,b)t^m x^a y^b$$

The recursion becomes a set of equations in the F_θ. eg. for spiral walks,

$$\begin{aligned}
F_e(x,y) &= tx + txF_e(x,y) + txF_s(x,y) \\
F_n(x,y) &= ty + tyF_e(x,y) + tyF_n(x,y) \\
F_w(x,y) &= t\overline{x} + t\overline{x}F_n(x,y) + t\overline{x}F_w(x,y) \\
F_s(x,y) &= t\overline{y} + t\overline{y}F_w(x,y) + t\overline{y}F_s(x,y).
\end{aligned}$$

Equivalently,

$$(\mathbf{I} - \hat{\mathbf{T}}^\top) \cdot \begin{pmatrix} F_e(x,y) \\ F_n(x,y) \\ F_w(x,y) \\ F_s(x,y) \end{pmatrix} = \begin{pmatrix} tx \\ ty \\ t\overline{x} \\ t\overline{y} \end{pmatrix}$$

where

$$\hat{\mathbf{T}} = \mathbf{T} \cdot \begin{pmatrix} tx & 0 & 0 & 0 \\ 0 & ty & 0 & 0 \\ 0 & 0 & t\overline{x} & 0 \\ 0 & 0 & 0 & t\overline{y} \end{pmatrix}$$

So

$$\begin{pmatrix} F_e(x,y) \\ F_n(x,y) \\ F_w(x,y) \\ F_s(x,y) \end{pmatrix} = (\mathbf{I} - \hat{\mathbf{T}}^\top)^{-1} \cdot \begin{pmatrix} tx \\ ty \\ t\overline{x} \\ t\overline{y} \end{pmatrix}$$

What are the numerator and denominator of F_θ? Want a more combinatorial construction. Write

$$F_\theta(t;x,y) = A_\theta(t;x,y) + B_\theta(t;x,y)F_\theta(t;x,y)$$

where A_θ counts walks with no θ steps except for the last step, and B_θ counts the subset of those which can follow a θ step. So

$$\begin{aligned} F_\theta(x,y) &= \frac{A_\theta(x,y)}{1 - B_\theta(x,y)} \\ &= A_\theta(x,y) + A_\theta(x,y)B_\theta(x,y) + A_\theta(x,y)B_\theta(x,y)^2 + \ldots \end{aligned}$$

A_θ and B_θ have simple solutions involving $\hat{\mathbf{T}}$.

Define

- $\mathbf{I}_\theta$ to be $\mathbf{I}$ with 0 at the θ-th position on the diagonal instead of 1;
- V_θ to be the 1×4 vector with 1 in the θ-th position and 0 elsewhere;
- $\hat{\mathbf{T}}_{*\theta}$ (resp. $\hat{\mathbf{T}}_{\theta *}$) to be the θ-th column (resp. row) of $\hat{\mathbf{T}}$.

Then

$$A_\theta(x, y) = \left(V_\theta + (\hat{\mathbf{T}}_{*\theta})^\top \cdot (\mathbf{I} - \mathbf{I}_\theta \hat{\mathbf{T}}^\top)^{-1} \mathbf{I}_\theta \right) \cdot \begin{pmatrix} tx \\ ty \\ t\overline{x} \\ t\overline{y} \end{pmatrix}$$

$$B_\theta(x, y) = (\hat{\mathbf{T}}_{*\theta})^\top \cdot \left(V_\theta + (\mathbf{I} - \mathbf{I}_\theta \hat{\mathbf{T}}^\top)^{-1} \mathbf{I}_\theta \cdot (\hat{\mathbf{T}}_{\theta *})^\top \right)$$

Asymptotics

Jordan decomposition of **T** leads to

$$c_m = c_1 \cdot \mathbf{S}\mathbf{J}^m\mathbf{S}^{-1}$$

where **J** is Jordan normal form of **T** and **S** is matrix of generalised eigenvectors.

Since **T** is primitive, by Perron-Frobenius it has a unique eigenvalue of greatest magnitude – call this μ. Assume WLOG that $\mathbf{J}_{1,1} = \mu$. Then

$$c_m \sim \|\mathbf{S}_{*1}\|_1 \times \mu^{m-1} \times (\mathbf{S}^{-1})_{1*},$$

where $\|\mathbf{S}_{*1}\|_1$ is the sum of the first column of **S** (ie. the eigenvector of **T** corresponding to μ), and $(\mathbf{S}^{-1})_{1*}$ is the first row of $\mathbf{S}^{-1}$.

$\rho = 1/\mu$ is the smallest positive root of $B_\theta(t; 1, 1) = 1$ (this is independent of θ).

Location of the endpoint and drift

Two possible behaviours for drift in the $\mathbf{x}$ direction. Let $\langle \mathbf{x}_m \rangle$ be the average endpoint $\mathbf{x}$-coordinate of walks of length m.

Lemma

If $B_\theta^{(0,1,0)}(\rho; 1, 1) \neq 0$ then $\langle \mathbf{x}_m \rangle$ behaves asymptotically as

$$\langle \mathbf{x}_m \rangle \sim \delta_{\mathbf{x}} \cdot m \qquad \text{as } m \to \infty,$$

where

$$\delta_{\mathbf{x}} = \frac{B_\theta^{(0,1,0)}(\rho; 1, 1)}{\rho B_\theta^{(1,0,0)}(\rho; 1, 1)}.$$

If instead $B_\theta^{(0,1,0)}(\rho; 1, 1) = 0$ then

$$\langle \mathbf{x}_m \rangle \sim \left(-\frac{B_\theta^{(1,1,0)}(\rho; 1, 1)}{B_\theta^{(1,0,0)}(\rho; 1, 1)} + \frac{A_\theta^{(0,1,0)}(\rho; 1, 1)}{A_\theta(\rho; 1, 1)} \right).$$

In addition

$$\delta_{\mathbf{x}} = \frac{d}{dx} \log \mu(x, 1) \Big|_{x=1}$$

where $\mu(x, y)$ is the smallest positive root of $B(t; x, y) = 1$.

Analogous for drift in $\mathbf{y}$ direction.

Upper half-plane

To rule out 'uninteresting' models in the upper half-plane, need another classification.

Definition

A (aperiodic) rule $\mathcal{R}$ is vertically unbounded if a walk obeying $\mathcal{R}$ can take two north steps without a south step in between, and two south steps without a north step in between.

In the upper half-plane, fewer symmetries to exploit: can only swap east and west steps.
Burnside's Lemma again:

Proposition

In the upper half-plane there are 9722 non-isomorphic, vertically unbounded rules.

Generating functions

Upper half-plane partition functions $\hat{\theta}_m^+(x, y)$ for each step direction, with generating functions $H_\theta(x, y) \equiv H_\theta(t; x, y)$. They satisfy the system

$$(\mathbf{I} - \hat{\mathbf{T}}^\top) \cdot \begin{pmatrix} H_e(x,y) \\ H_n(x,y) \\ H_w(x,y) \\ H_s(x,y) \end{pmatrix} = \begin{pmatrix} tx \\ ty \\ t\bar{x} \\ 0 \end{pmatrix} - (\mathbf{I} - \mathbf{I}_s) \cdot \hat{\mathbf{T}} \cdot \begin{pmatrix} H_e(x,0) \\ H_n(x,0) \\ H_w(x,0) \\ H_s(x,0) \end{pmatrix}$$

Again, we seek a more combinatorially 'clear' form:

$$H_\theta(x, y) = C_\theta(x, y) + B_\theta(x, y) H_\theta(x, y) - D_\theta(x, y) H^*(x)$$

where

- $C_\theta(x, y) \equiv C_\theta(t; x, y)$ is the GF of walks counted by A_θ (ie. in the full plane) which do not start with a south step
- $D_\theta(x, y) \equiv D_\theta(t; x, y)$ is the GF of walks counted by A_θ which do start with a south step
- $H^*(x) \equiv H^*(t; x)$ is the GF of all walks (in the upper half-plane) which (a) end on the surface, and (b) end with a step that can precede a south step.

C_θ and D_θ have simple solutions like A_θ and B_θ.

So

$$H_\theta(x,y) = \frac{C_\theta(x,y) - D_\theta(x,y)H^*(x)}{1 - B_\theta(x,y)}.$$

Need solution to $H^*(x)$!

Rearrange to kernel form:

$$(1 - B_\theta(x,y))H_\theta(x,y) = C_\theta(x,y) - D_\theta(x,y)H^*(x)$$

The kernel has two roots $\upsilon^-(t;x)$ and $\upsilon^+(t;x)$. The smaller is a power series in t.

Proposition

For a vertically unbounded rule, let $\upsilon^-(t;x)$ *be the smaller of the two functions satisfying* $B_\theta(t;x,\upsilon^\pm(t;x)) = 1$ *for any of the* θ*. Then*

$$H^*(x) = \frac{C_\theta(x,\upsilon^-(t;x))}{D_\theta(x,\upsilon^-(t;x))}.$$

Asymptotics

For vertically unbounded rules the generating functions are always algebraic (and not rational) but the asymptotics depend on the vertical drift.

Theorem

When $\delta_{\mathbf{y}} > 0$, the dominant singularity is a simple pole at $t = \rho$ and

$$\theta_m^+ \sim c_\theta^+ \mu^m$$

for an easily computable algebraic number c_θ^+.

Theorem

When $\delta_{\mathbf{y}} = 0$, the dominant singularity is the reciprocal of a square root at $t = \rho$ and

$$\theta_m^+ \sim \frac{c_\theta^0}{\sqrt{\pi m}} \mu^m$$

for an easily computable algebraic number c_θ^0.

Theorem

When $\delta_{\mathbf{y}} < 0$, the dominant singularity is a square root at $t = \kappa$, one of the singularities of $\upsilon^-(t; 1)$ (effectively a root of the discriminant of $1 - B_\theta$). With $\lambda = 1/\kappa$, we have

$$\theta_m^+ \sim \frac{c_\theta^-}{2\sqrt{\pi} m^{3/2}} \lambda^m$$

with c_θ^- an easily computable algebraic number.

Note that $\lambda < \mu$.

With a bit of work the average endpoint location can be computed in all cases.

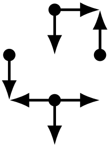
Quarter plane

The only symmetry is reflection in the line $\mathbf{y} = \mathbf{x}$. The conditions for a model to be 'interesting' are more complicated.

First, insist on models being vertically and horizontally ("cardinally") unbounded: can take two θ steps without a $-\theta$ step in between. Up to symmetry there are 7520 such models.

This is not enough though. Consider

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$



Walks obeying this cannot step above $\mathbf{y} = \mathbf{x} + 1$. So almost every half plane walk is also a quarter plane walk (exceptions are those which visit $(-1, 0)$).

To exclude these, define $\mathbf{T}$ to be south-east bound if all the following are zero:

$$\mathbf{T}_{nn} = \mathbf{T}_{ww} = \mathbf{T}_{nw}\mathbf{T}_{wn} = \mathbf{T}_{ne}\mathbf{T}_{ew}\mathbf{T}_{wn} = \mathbf{T}_{nw}\mathbf{T}_{we}\mathbf{T}_{en} = \mathbf{T}_{wn}\mathbf{T}_{ns}\mathbf{T}_{sw} = \mathbf{T}_{ws}\mathbf{T}_{sn}\mathbf{T}_{nw} = 0.$$

Similarly, $\mathbf{T}$ is north-west bound if

$$\mathbf{T}_{ee} = \mathbf{T}_{ss} = \mathbf{T}_{es}\mathbf{T}_{se} = \mathbf{T}_{en}\mathbf{T}_{ns}\mathbf{T}_{se} = \mathbf{T}_{es}\mathbf{T}_{sn}\mathbf{T}_{ne} = \mathbf{T}_{se}\mathbf{T}_{ew}\mathbf{T}_{ws} = \mathbf{T}_{sw}\mathbf{T}_{we}\mathbf{T}_{es} = 0.$$

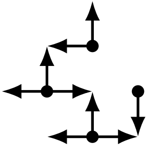
and south-west bound if

$$\mathbf{T}_{ee} = \mathbf{T}_{nn} = \mathbf{T}_{ne}\mathbf{T}_{en} = \mathbf{T}_{en}\mathbf{T}_{ns}\mathbf{T}_{se} = \mathbf{T}_{es}\mathbf{T}_{sn}\mathbf{T}_{ne} = \mathbf{T}_{ne}\mathbf{T}_{ew}\mathbf{T}_{wn} = \mathbf{T}_{nw}\mathbf{T}_{we}\mathbf{T}_{en} = 0.$$

Want to exclude all these ("diagonally unbounded"). Up to symmetry, there are 7146 models which are cardinally and diagonally unbounded.

Still not enough. Consider

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



Walks obeying this can never leave the **x**- or **y**-axes.

Need one more definition: $\mathbf{T}$ is glued if

$$\mathbf{T}_{en} = \mathbf{T}_{ee}\mathbf{T}_{ew}\mathbf{T}_{wn} = \mathbf{T}_{ne} = \mathbf{T}_{nn}\mathbf{T}_{ns}\mathbf{T}_{se} = 0.$$

Proposition

In the first quadrant there are 6909 non-isomorphic two-step rules which are cardinally and diagonally unbounded and unglued.

Functional equations

Significantly more complicated. Generating functions satisfy the system

$$(\mathbf{I}-\hat{\mathbf{T}}^\top)\cdot\begin{pmatrix}Q_e(x,y)\\Q_n(x,y)\\Q_w(x,y)\\Q_s(x,y)\end{pmatrix}=\begin{pmatrix}tx\\ty\\0\\0\end{pmatrix}-(\mathbf{I}-\mathbf{I}_s)\cdot\hat{\mathbf{T}}\cdot\begin{pmatrix}Q_e(x,0)\\Q_n(x,0)\\Q_w(x,0)\\Q_s(x,0)\end{pmatrix}-(\mathbf{I}-\mathbf{I}_w)\cdot\hat{\mathbf{T}}\cdot\begin{pmatrix}Q_e(0,y)\\Q_n(0,y)\\Q_w(0,y)\\Q_s(0,y)\end{pmatrix}$$

which becomes

$$Q_\theta(x,y)=G_\theta(x,y)+B_\theta(x,y)Q_\theta(x,y)-D_\theta(x,y)Q^{\downarrow}(x)-J_\theta(x,y)Q^{\leftarrow}(y).$$

- B_θ and D_θ are as above
- $G_\theta(x,y)\equiv G_\theta(t;x,y)$ is the GF of walks counted by A_θ which start east or north
- $J_\theta(x,y)\equiv J_\theta(t;x,y)$ is the GF of walks counted by A_θ which start west
- $Q^{\downarrow}(x)\equiv Q^{\downarrow}(t;x)$ counts quarter-plane walks which (a) end on the x-axis and (b) end with a step type that can precede south. Similarly $Q^{\leftarrow}(y)\equiv Q^{\leftarrow}(t;y)$ for walks ending on y-axis.

Groups?

Rewrite as

$$(1 - B_\theta(x,y))Q_\theta(x,y) = G_\theta(x,y) - D_\theta(x,y)Q^{\downarrow}(x) - J_\theta(x,y)Q^{\leftarrow}(y).$$

Again B_θ, G_θ, D_θ, J_θ have simple rational solutions.

Cannot simply cancel the LHS à la the half-plane problem, because unknowns on the RHS depend on x and y.

Is there a group, like the "regular" lattice paths?

$B_\theta(x,y)$ plays the role of $tS(x,y)$ here. Can look for substitutions which leave B_θ unchanged?

Lemma

For any cardinally unbounded rule, the equation

$$B_\theta(x,Y) = B_\theta(x,y)$$

has two distinct solutions in Y, one of which is $Y = y$ and the other is a rational function of t, x, y. Similarly,

$$B_\theta(X,y) = B_\theta(x,y)$$

has two distinct solutions in X, one of which is $X = x$ and the other is a rational function of t, x, y.

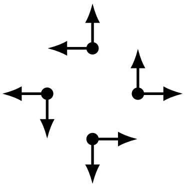
Let $X = \Psi_\theta(t; x, y)$ and $Y = \Phi_\theta(t; x, y)$ be the rational substitutions.

Easy to see that the operations $y \mapsto \Phi_\theta(t; x, y)$ and $x \mapsto \Psi_\theta(t; x, y)$ are involutions.

So we do get a group! But does it depend on θ?

Yes and no... the actual group elements depend on θ, but the order of the group (conjecturally) does not. eg. Spiral walks:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$



$$\Psi_e(t; x, y) = \frac{t(t^2 - tx - ty + xy + t^2xy + t^2y^2 - txy^2)}{(x-t)(y-t)(1-ty)} \qquad \Phi_e(t; x, y) = \overline{y}.$$

Group is

$$\mathcal{G}_e = \{(x, y), (\Psi_e, y), (\Psi_e, \overline{y}), (x, \overline{y})\}.$$

Meanwhile

$$\Psi_n(t; x, y) = \overline{x} \qquad \Phi_n(t; x, y) = \Psi_e(t; y, x).$$

with group

$$\mathcal{G}_n = \{(x, y), (\overline{x}, y), (\overline{x}, \Phi_n), (x, \Phi_n)\}.$$

and

$$\Psi_w(t; x, y) = -\frac{(y-t)(1-tx)(1-ty)}{t(t - t^2x - y - t^2y + txy + ty^2 - t^2xy^2)} \qquad \Phi_w(t; x, y) = \overline{y}.$$

with group

$$\mathcal{G}_w = \{(x, y), (\Psi_w, y), (\Psi_w, \overline{y}), (x, \overline{y})\}.$$

and

$$\Psi_s(t; x, y) = \overline{x} \qquad \Phi_s(t; x, y) = \Psi_w(t; y, x).$$

with group

$$\mathcal{G}_s = \{(x, y), (\overline{x}, y), (\overline{x}, \Phi_s), (x, \Phi_s)\}.$$

Note that Ψ_e and Φ_n are power series in t but Ψ_w and Φ_s are not.

Conjecture

The order of the group $\mathcal{G}_\theta$ is the same for all four directions θ. For the 6909 quarter plane models, the breakdown is

- *1084 have group isomorphic to D_2*
- *443 have group isomorphic to D_3*
- *146 have group isomorphic to D_4*
- *66 have group isomorphic to D_5*
- *6 have group isomorphic to D_6*
- *5164 have group isomorphic to D_∞*

An orbit sum

Using $\theta = e$, apply the four group symmetries and take a linear combination to eliminate $Q^{\downarrow}(x), Q^{\leftarrow}(y), Q^{\downarrow}(\Psi_e), Q^{\leftarrow}(\overline{y})$:

$$\frac{y(t-y)Q_e(x,y)}{1-ty} - \frac{t^3xy^2(t-y)Q_e(\Psi_e,y)}{(x-t)(1-ty)(t^2-tx-ty+xy+t^2xy+t^2y^2-txy^2)} - \frac{t^3xyQ_e(\Psi_e,\overline{y})}{(x-t)(t^2-tx-ty+xy+t^2xy+t^2y^2-txy^2)} + Q_e(x,\overline{y}) = \frac{RHS_e}{1-B_e(x,y)},$$

for a rational RHS_e.

Take $[x^> y^>]$ part of this, only the first term on the LHS survives.

Theorem

The generating function $Q_e(x,y)$ for spiral walks has the solution

$$Q_e(x,y) = -\frac{\overline{y}(1-ty)}{y-t}[x^> y^>]\left(\frac{RHS_e}{1-B_e(x,y)}\right)$$
$$= tx + t^2x^2 + t^3x^3 + t^4x^4 + t^5(x+x^5) + t^6(2x^2+x^6+xy) + O(t^7).$$

Being the positive part of a rational function, $Q_e(x,y)$ is D-finite.

This process can (essentially) be repeated to obtain $Q_n(x, y)$.

Cannot be repeated for $\theta = w$ or $\theta = s$.

However, note that $Q^{\leftarrow}(0) = 0$. So

- take $y = 0$ to get an equation between $Q_\theta(x, 0)$ and $Q^{\downarrow}(x)$ for each θ
- use the above solution with $\theta = e$ to solve $Q^{\downarrow}(x)$
- substitute back in to solve $Q^{\leftarrow}(y)$
- use these to solve $Q_w(x, y)$ and $Q_s(x, y)$.

No rigorous asymptotics (yet) but

Conjecture

For spiral walks in the quarter plane,

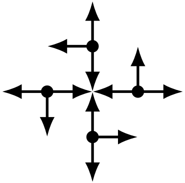
$$P_m^{\llcorner} = \frac{8}{\pi} \times \frac{1}{m} \times 2^m \times \left(1 - \frac{3}{2m} + \mathrm{O}\left(\frac{1}{m^2}\right)\right).$$

Similar calculations for walks ending on the boundaries or at the corner.

The (many) cases where orbit sums fail

(1): Finite group but cannot cancel all unknowns on the RHS:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

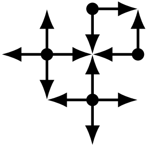


Similar group structure to spirals (group D_2, series in t for $\theta = e, n$ but not w, s).

Analysis of 3000 term series turns up nothing – almost certainly not D-finite.

(2): Orbit sum completely vanishes:

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



Group is isomorphic to D_3

$$G_e = G_n = \{(x,y),(\psi,y),(y,\psi),(y,x),(\psi,x),(x,\psi)\}$$

where

$$\psi = \frac{t(1+xy)}{xy - tx - ty - t^2xy}.$$

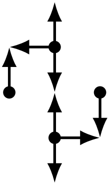
Vanishing orbit sum is reminiscent of the four algebraic "regular" lattice path models. Indeed 50 terms are enough to guess

$$-t(2+4t-19t^2-22t^3-9t^4) + 2(1+t)(1-16t^2+16t^3+18t^4)Q_e + t(5-16t-24t^2+64t^3+54t^4)Q_e^2 + 4t^2(1+t)(1-3t)^2Q_e^3 + t^3(1-3t)^2Q_e^4 = 0.$$

But have not been able to get half orbit sum to work here.

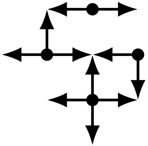
(3): Infinite group, but with a D-finite generating function:

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$



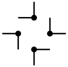
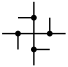
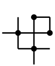
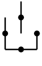
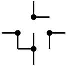
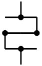
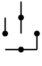
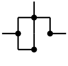
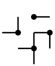
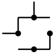
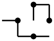
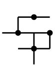
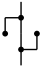
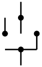
(4): Infinite group, but with an algebraic generating function:

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



Computational results

Using Ore Algebra package for Sage (Kauers, Jaroschek, Johansson 2015+) to analyse ≥ 500 terms for each model.

	alg (lower bound)	Df (?)	nonDf (upper bound)
D_2	–	659 	425 
D_3	40 	66 	337 
D_4	5 	59 	82 
D_5	6 	4 	56 
D_6	–	–	6 
D_∞	22 	30 	5112 

With more terms, almost certainly some non-D-finite will become D-finite and some D-finite will become algebraic.

Conclusion & open questions

- Everything straightforward in full and half plane, but functions $B_\theta(x, y)$ etc provide useful framework for working in quarter plane.
- Each model has a group but the combinatorial meaning is very unclear
  - Can the group(s) be understood in terms of the matrix $\mathbf{T}$?
  - Or at least prove that $|\mathcal{G}_\theta|$ is independent of θ?
  - Clearly the group is not as closely tied to properties of $Q(x, y)$ as for “regular” lattice paths.
- Does the drift have some influence on the solvability?
- Many other methods which might be useful here.

arXiv:2010.06955

github.com/nrbeaton/two_step_paths

Thank you!