Chemical distance for the half-orthant model

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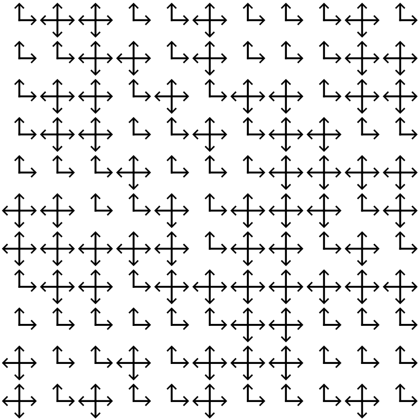
Introduction & definitions

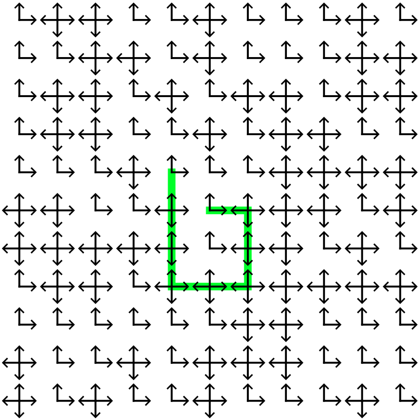
The half-orthant model is a lattice model of a random environment. Let $\{e_1, \ldots, e_d\}$ be the canonical basis vectors. Independently at each site $x \in \mathbb{Z}^d$:

- with probability p insert arrows from x to each neighbour $x + e_i$, $i = 1, \ldots, d$;
- otherwise (with probability $1 - p$) insert arrows from x to all neighbours $x \pm e_i$.



Let $\boldsymbol{\omega} = (\omega_x)_{x \in \mathbb{Z}^d}$ be such an environment and let $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ be a nearest-neighbour path.

γ is consistent with $\boldsymbol{\omega}$ if for each step $\gamma_j - \gamma_{j-1}$, $\omega_{\gamma_{j-1}}$ contains the arrow pointing towards γ_j.



Let $\mathcal{C}_o(p)$ be the set of points reachable from the origin by following arrows.

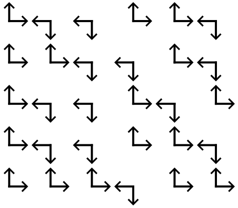
- $C_o(1)$ is the positive orthant $\mathbb{O}_+ = \{x \in \mathbb{Z}^d : x^{[i]} \geq 0 \text{ for all } i = 1, \ldots, d\}$
- $C_o(0)$ is the whole space $\mathbb{Z}^d$
- There is a natural coupling so that $C_o(p)$ is decreasing in p

Theorem (Holmes & Salisbury 2021)

There exists a critical value $p_c(d) \in (0.57, 1)$ such that a.s. $\mathcal{C}_o(p) = \mathbb{Z}^d$ if $p < p_c(d)$, and a.s. $\mathcal{C}_o(p) \neq \mathbb{Z}^d$ if $p > p_c(d)$.

Aside: there is also the orthant model, where for each site x

- with probability p insert arrows from x to each $x + e_i$, $i = 1, \ldots, d$
- with probability $1 - p$ insert arrows from x to each $x - e_i$, $i = 1, \ldots, d$.



For p large there is a shape theorem for C_o [H&S 2021], but for $d \geq 3$ this does not extend all the way to $p > p_c(d)$. (Essentially $C_o(p)$ looks like a cone.)

Here we consider a different kind of shape theorem which depends on the chemical distance.

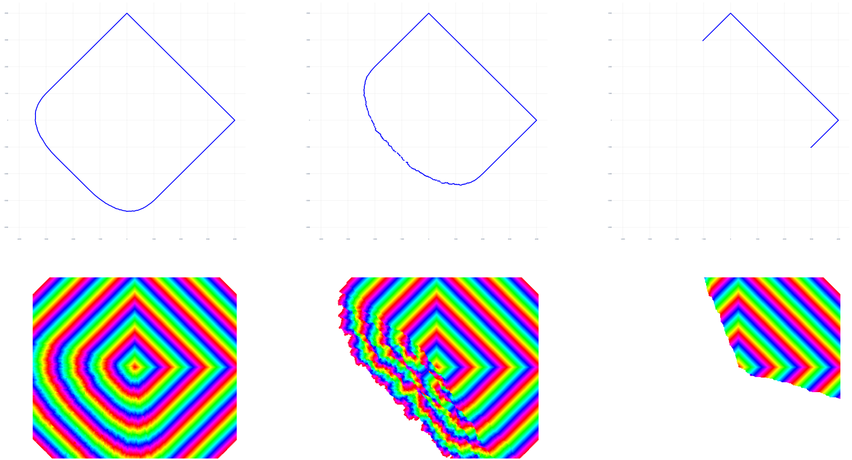
For sites u, v and an environment $\boldsymbol{\omega}$, let $T_{u,v}$ be the length of any consistent shortest path from u to v. (If there is no path, set $T_{u,v} = \infty$.) This is the chemical distance from u to v. Also define $T_v = T_{o,v}$.

$\rightarrow$ Related to first passage percolation.

If $p < p_c(d)$ then $C_o(p) = \mathbb{Z}^d$ a.s., so $T_v < \infty$. Translation invariance $\Rightarrow T_{u,v} < \infty$ a.s.

Define $A_n = \{x \in \mathbb{Z}^d : T_x = n\}$.

- If $p = 0$ then A_n is the ℓ_1 ball $B_n = \{x \in \mathbb{Z}^d : \|x\|_1 = n\}$.
- If $p = 1$ then A_n is $B_n^+ = \{x \in \mathbb{O}^+ : \|x\|_1 = n\}$.



Top row: Simulation of the set of points whose distance from the origin within the random environment is exactly $n = 4000$, for $p = 0.25$, $p = 0.5$, $p = 0.75$ (left to right) respectively. Bottom row: Simulations of these sets for different values of n, for $p = 0.25$, $p = 0.5$, $p = 0.75$.

Theorem 1: Law of large numbers for the chemical distance

Let $p_*(d)$ be the critical parameter for site percolation on $\mathbb{Z}^d$, and $p_\dagger(d)$ be the critical parameter for oriented site percolation on $\mathbb{Z}^d$.

Theorem 1 (BHH 2024)

Fix $d \geq 2$. For p such that $1 - p > \min\{1 - p_c(2), p_*(d)\}$ there exists a function $\zeta_p : \mathbb{Z}^d \to \mathbb{R}_+$ such that for each $v \in \mathbb{Z}^d$

$$\lim_{n\to\infty} \frac{T_{nv}(p)}{n} = \zeta_p(v), \quad \textit{almost surely and in } L^1.$$

Trivial for $v = o$ with $\zeta_p(o) = 0$.

For other v, follows from an application of the subadditive ergodic theorem.

The difficulty is in proving that expected passage times are finite.

Subadditive ergodic theorem

Fix $v \in \mathbb{Z}^d \setminus \{o\}$ and let $X_{m,n} = T_{mv,nv}$ be the chemical distance from mv to nv.

Theorem (Kingman, Liggett)

Let $(X_{m,n})_{0 \le m < n}$ be a family of random variables satisfying

- **(i)** $X_{0,n} \le X_{0,k} + X_{k,n}$ *for all* $0 < k < n$;
- **(ii)** *the distribution of the sequence* $(X_{m,m+k})_{k \ge 1}$ *does not depend on* m;
- **(iii)** *for each* n, $\mathbb{E}[|X_{0,n}|] < \infty$ *and* $\mathbb{E}[X_{0,n}] \ge cn$ *for some constant* $c > -\infty$;
- **(iv)** *for each* k, $(X_{nk,(n+1)k})_{n \ge 1}$ *is a stationary and ergodic sequence.*

Then $X = \lim_{n \to \infty} \frac{X_{0,n}}{n}$ *exists a.s. and in* L^1, *and* X *is deterministic.*

We show that our $X_{m,n} = T_{mv,nv}$ satisfies these properties:

- (ii) and (iv) follow from the fact that the environment is i.i.d.
- (i) is straightforward: if γ' is a consistent path from o to kv, and γ'' is a consistent path from kv to nv, then the concatenation $\gamma = \gamma' \circ \gamma''$ is a constent path from o to nv
- trivially $\mathbb{E}[T_{o,nv}] \ge 0$ since $T_{o,nv} \ge 0$

Remains to show $E[|T_{o,nx}|] < \infty$ for every x.

Finite expected passage times

Define $\mathcal{E} := \{\pm e_i : i \in [d]\}$

Every $x \in \mathbb{Z}^d$ is a sum of $\|x\|_1$ vectors $e \in \mathcal{E}$, so it suffices to show that $\mathbb{E}[T_{o,e}] < \infty$ for each $e \in \mathcal{E}$.

For the half-orthant model this is trivial if e is in the first orthant. Moreover by symmetry $\mathbb{E}[T_{o,-e_j}] = \mathbb{E}[T_{o,-e_1}]$ so we only need to show $\mathbb{E}[T_{o,-e_1}] < \infty$.

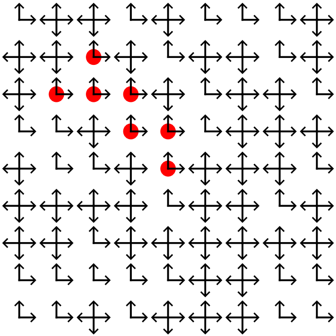
Finite expected passage times: $p < p_c(2)$

Only need to show it is true for $d = 2$, i.e. using paths that stay in $\mathbb{Z}^2 \times \{0\}^{d-2}$.

Define a cluster of sites $\mathfrak{C}$ as follows:

- if the origin is a full site, $\mathfrak{C} = \{o\}$
- else $\mathfrak{C}_0 = \{o\}$, and then $\mathfrak{C}_n$ is defined by

$$x \in \mathfrak{C}_n \text{ if at least one of } \begin{cases} x + e_1 & \in \mathfrak{C}_{n-1} \\ x - e_2 & \in \mathfrak{C}_{n-1} \\ x + e_1 - e_2 & \in \mathfrak{C}_{n-2} \end{cases}$$



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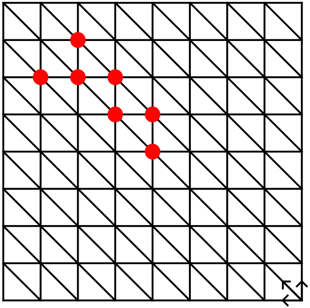
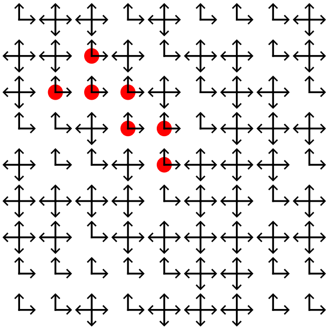


$\mathfrak{C}$ is the cluster of the origin for an oriented site percolation on the triangular lattice.

$p_c(2) \approx 0.59$ is also the critical point for oriented site percolation on the triangular lattice [Holmes & Salisbury 2014], so if $p < p_c(2)$ then $\mathfrak{C}$ is finite.

Lemma

There is a consistent path from o to $-e_1$ of length at most $4|\mathfrak{C}|$.



Proof by induction on $|\mathfrak{C}|$. If $|\mathfrak{C}| = 1$ there is a path from o to e_{-1} of length 1 or 3.

Otherwise say $|\mathfrak{C}| = n + 1$. There is always a point $x \in \mathfrak{C}$ such that $x + e_2, x - e_1, x + e_2 - e_1 \notin \mathfrak{C}$.

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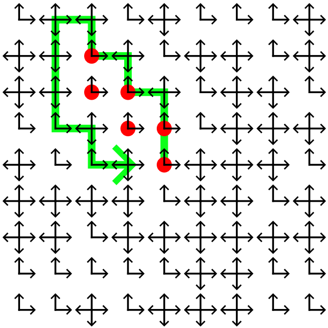

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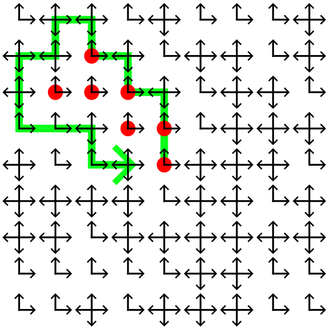

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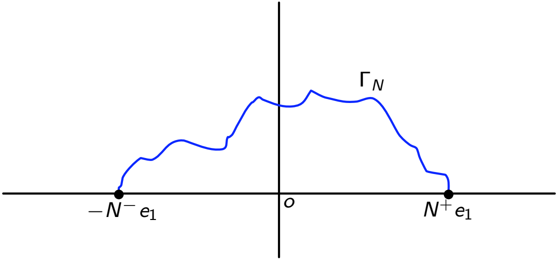
Now switching x back to a half site increases the length of the path by at most 4.

Finite expected passage times: $1 - p > p_*(d)$

If $q := 1 - p > p_*(d)$ then the full sites percolate (ignoring half sites). Let $\mathcal{P}$ be the unique infinite cluster.

Define $N^+ = \inf\{n \geq 0 : ne_1 \in \mathcal{P}\}$ and $N^- = \inf\{n > 0 : -ne_1 \in \mathcal{P}\}$.

Let Γ_N be any shortest path from N^+e_1 to $-N^-e_1$ and let $M = |\Gamma_N|$.



If $N^+ + M + N^-$ has finite expectation then we are done.

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Lemma (Grimmett 2023, private communication)

For $q > p_*(d)$ there exist $c, C > 0$ such that for all $n > 0$,

$$\mathbb{P}_q^*(N^+ > n) \leq Ce^{-cn}.$$

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Let $\mathbb{P}_p^b$ denote the law of bond percolation on $\mathbb{Z}^d$, and let $p_b(d)$ be the critical parameter. Let $D_b(u, v)$ be the shortest path distance between u and v using open bonds.

Theorem (Antal & Pisztora 1996)

For $p > p_b(d)$, there exists a constant $\rho(p, d) \geq 1$ such that

$$\limsup_{|y|\to\infty} \frac{1}{|y|} \log \mathbb{P}_p^b(0 \leftrightarrow y, D(o, y) > \rho|y|) < 0.$$

Moreover for $p > p_b(d)$ and for any $y \in \mathbb{Z}^d$,

$$\limsup_{\ell\to\infty} \frac{1}{\ell} \log \mathbb{P}_p^b(0 \leftrightarrow y, D(o, y) > \ell) < 0.$$

Can be shown to also hold for site percolation.

Extensions & consequences of Theorem 1

We defined $\zeta_p(v) : \mathbb{Z}^d \to \mathbb{R}_+$ by

$$\zeta_p(v) = \lim_{n\to\infty} \frac{T_{nv}(p)}{n}.$$

Easy to extend to $\mathbb{Q}^d$:

$$\zeta_p(u) = \frac{\zeta_p(m_u u)}{m_u}$$

where $m_u = \inf\{m \in \mathbb{N} : mu \in \mathbb{Z}^d\}$.

For $y \in \mathbb{R}^d$ define $[y] \in \mathbb{Z}^d$ to be the unique lattice point s.t. $y \in [y] + [0,1)^d$.

Can then extend Theorem 1 by taking a limit through $\mathbb{R}$ instead of $\mathbb{Z}$:

Lemma 1

Take $1 - p > \min\{1 - p_c(2), p_*(d)\}$ *and* $u \in \mathbb{Q}^d$. *Then*

$$\lim_{x\to\infty} \frac{T_{[xu]}}{x} = \zeta_p(u), \quad \textit{a.s. and in } L^1,$$

where the limit is through $\mathbb{R}$.

Then for $v \in \mathbb{Z}^d \setminus \{o\}$ define

$$D_n(v, p) = \sup\{k \in [0, \infty) : T_{[kv]}(p) \leq n\},$$

i.e. how far can we go in direction v before reaching a point more than n steps from the origin?

Corollary 1

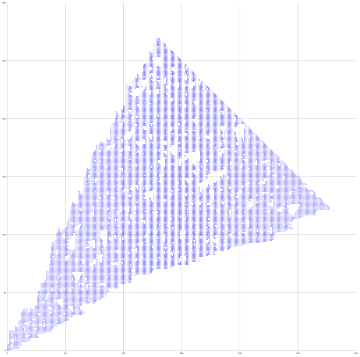
For $1 - p > \min\{1 - p_c(2), p_*(d)\}$ *and* $v \in \mathbb{Z}^d \setminus \{o\}$,

$$\lim_{n\to\infty} \frac{D_n(v,p)}{n} = \frac{1}{\zeta_p(v)} \quad \textit{almost surely and in } L^1.$$

Theorem 2: Features of the shape

Let $O \subset \mathcal{E}$ be a basis for $\mathbb{Z}^d$ (i.e. $|d|$ orthogonal vectors). Then consider oriented site percolation with orientation given by O. Each choice of O is identical in law except for change of orientation.

With $q > p_*(d)$ (supercrtitical), conditional on the cluster of the origin being infinite, the cluster of the origin is a cone with axis $\sum_{e \in O} e$. Call this cone O_q.



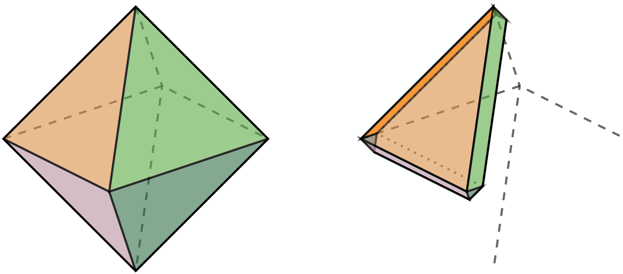
Define

$$\mathcal{O} = \{O \subset \mathcal{E} : O \text{ is a basis for } \mathbb{Z}^d\}$$

Let $\mathbb{D} = \mathbb{Q}^d \setminus \{o\}$ and define

$$\mathcal{S}_p^{\text{good}} = \mathbb{D} \cap \left\{ \sum_{i=1}^{d} (\alpha_i - \beta_i) e_i : \quad a \in [p, 1], \quad 0 \le \alpha_i, \beta_i \ \forall i \in [d], \text{ and } \right.$$
$$\left. \sum_{i=1}^{d} \alpha_i = a = 1 - \sum_{i=1}^{d} \beta_i, \text{ and } \alpha_i \beta_i = 0 \text{ for each } i \in [d] \right\}.$$

$\mathcal{S}_p^{\text{good}}$ is a subset of the boundary of the ℓ_1 sphere, and contains the section of this boundary in the first orthant:



Let $\mathcal{S}_p = \{u \in \mathbb{D} : \zeta_p(u) = \|u\|_1\}$. That is, $\mathcal{S}_p$ contains the directions where the chemical distance is a.s. equal to the ℓ_1 distance.

Theorem 2 (BHH 2024)

For $1 - p > \min\{1 - p_c(2), p_*(d)\}$, the function $\zeta_p : \mathbb{Q}^d \to \mathbb{R}$ satisfies

(a)
- **(i)** *for $q \in \mathbb{Q}_+$ and $u \in \mathbb{Q}^d$, $\zeta_p(qu) = q\zeta_p(u)$*
- **(ii)** $\zeta_p(u + v) \leq \zeta_p(u) + \zeta_p(v)$
- **(iii)** *ζ_p is uniformly continuous*

(b) *For fixed v, $\zeta_p(v)$ is non-decreasing in p.*

(c) *For $u \in \mathbb{D}$, $u \in \mathcal{S}_p$ when*
- **(i)** $u \in \mathcal{S}_p^{\mathrm{good}}$
- **(ii)** *$1 - p > p_\dagger(d)$ and $u \in \bigcup_{O \in \mathcal{O}} O_{1-p}$.*

(d) *For $u \in \mathbb{D}$ with $\|u\|_1 = 1$ we have $u \notin \mathcal{S}_p$ when*
- **(i)** *for any $i = 1, \ldots, d$, $\|u + e_i\|_1 < \epsilon$ for ϵ sufficiently small depending on d and p*
- **(ii)** *$d = 2$ and $u = (-(1 - s), s)$ for $s \in [0, p) \cap \mathbb{Q}$.*

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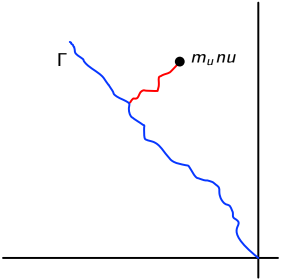
(a) and (b) straightforward.

Theorem 2 (c): directions where chemical distance $= \ell_1$ distance

For Theorem 2 (c)(i), $u \in \mathcal{S}_p^{\mathrm{good}}$, i.e. u is 'close' to the positive orthant.

Idea: Construct a (random) path Γ along which every point $x \in \Gamma$ is reachable in time $\|x\|_1$, and such that $m_u nu$ is reachable by following Γ for a long time and then taking $+e_i$ steps for a relatively short time.

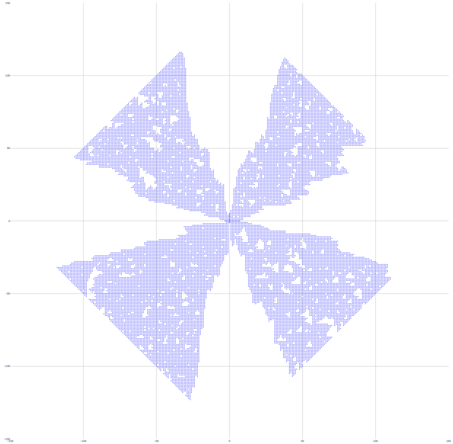


More precisely, we show that there is a constant C such that a.s. for each $\epsilon > 0$,

$$T_{m_u nu} \leq nm_u + C\epsilon n \quad \text{infinitely often.}$$

For Theorem 2(c)(ii), we have $1 - p > p_{\dagger}(d)$ and $u \in \bigcup_{O \in \mathcal{O}} O_{1-p}$.

That is, full sites percolate in the sense of oriented site percolation, in the direction of each of the 2^d orthants.

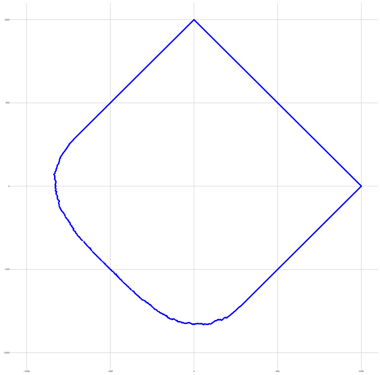


Lemma

Fix $d \geq 2$ and let $q > p_\dagger(d)$. For $v \in \mathbb{Z}^d$ in the interior of O_q,

$$\mathbb{P}_q^\dagger(o \to nv \text{ infinitely often}) \geq \inf_{n \in \mathbb{Z}_+} (o \to nv) = \eta(q) > 0$$

It follows that for such a v, $T_{nv} = \|nv\|_1$ infinitely often, and hence $\frac{T_{nv}}{n} \to \zeta_p(v)$, i.e. $v \in \mathcal{S}_p$.



Theorem 2 (d): directions where chemical distance $> \ell_1$ distance

For Theorem 2 (d)(i), u is close to $-e_i$ and $\zeta_p(u) < 1$.

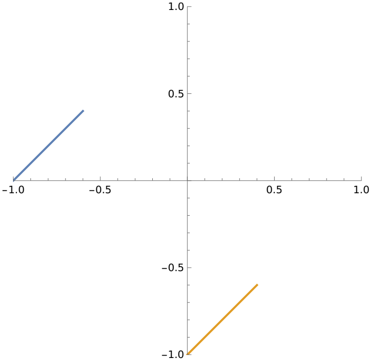
Basic idea: suppose $\|u\|_1 = 1$ and that we can reach un in at most $(1+\epsilon)n$ steps. Let D_n be this event.

We show that $\mathbb{P}(D_n)$ is summable in n by bounding the number of self-avoiding walks from o to un. The exponential growth rate can be made close to 1 by making ϵ small.

Then the probability that any of those SAWs is consistent with $\boldsymbol{\omega}$ decreases exponentially in n.

For ϵ sufficiently small, $\mathbb{P}(D_n)$ decreases exponentially. Hence by Borel-Cantelli, $\mathbb{P}(D_n \text{ infinitely often}) = 0$.

For Theorem 2 (d)(ii) we have $d = 2$ and $u = (-(1-s), s)$ for $s \in [0, p) \cap \mathbb{Q}$.



For Theorem 2 (d)(i) we controlled the growth rate of SAWs to u near $-e_1$ and made it close to 1.

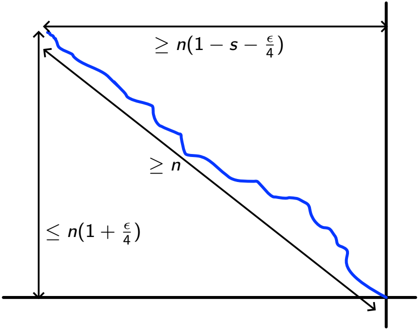
Here we cannot do that, since the number of SAWs from o to $(-(1-s)n, sn)$ of length n is $\binom{n}{sn}$, with growth rate

$$\frac{1}{(1-s)^{1-s}s^s} \in [1, 2]$$

Let $b(\gamma)$ be the number of $\rightarrow, \downarrow$ steps in γ. Let $\epsilon \in (0, \frac{p-s}{10})$. Note that if $\ell(\gamma) \leq n(1+\epsilon)$ and $b(\gamma) \geq \epsilon n$ then γ cannot reach the NW side of the ℓ_1 ball $B_1(0,n)$.

Define

$$J_n = \big\{\exists \gamma \text{ consistent with } \boldsymbol{\omega},\ \ell(\gamma) \leq n(1+\epsilon),\ b(\gamma) \leq \epsilon n \text{ such that}$$
$$\gamma_\ell^{(1)} \leq -n(1-s-\tfrac{\epsilon}{4}),\ 0 \leq \gamma_\ell^{(2)} \leq n(s+\tfrac{\epsilon}{4}),\ \|\gamma_\ell\|_1 \geq n\big\}$$



We prove that $\mathbb{P}(J_n) \leq Ce^{-cn}$ for some $C, c > 0$. So $\mathbb{P}(J_n)$ is summable $\Rightarrow$ Borel-Cantelli $\Rightarrow$ J_n does not occur infinitely often $\Rightarrow$ $\zeta_p(u) > 1$ in this direction.

Get a sequence of upper bounds on $\mathbb{P}(J_n)$:

1. Relax the consistency constraint so that only $\leftarrow$ steps must be consistent with $\boldsymbol{\omega}$
2. Insist that the path never steps $\uparrow$ when a $\leftarrow$ step is available
3. Remove any loops to get a self-avoiding walk ending at the same point
4. Construct a random SAW Γ whose (i.i.d.) increments have the same distribution as the vertices of $\boldsymbol{\omega}$
5. Relax the self-avoiding constraint
6. Relax the constraints that $\Gamma_\ell^{(1)} \leq -n(1-s-\frac{\epsilon}{4})$ and $\|\Gamma_\ell\|_1 \geq n$

Finally get

$$\mathbb{P}(J_n) \leq \sum_{\ell=n}^{n(1+\epsilon)} \sum_{b=0}^{\epsilon n} \sum_{\vec{t}} \sum_{\vec{x}} \mathbb{P}\left(\overline{\Gamma}_\ell^{(2)} \leq n(s+\tfrac{\epsilon}{4})\right)$$

where $\overline{\Gamma}$

- takes $\rightarrow, \downarrow$ steps at times $\vec{t} = (t_1, \ldots, t_b)$ in directions $\vec{x} \in \{\rightarrow, \downarrow\}^b$,
- at other times steps $\uparrow$ with probability p and $\leftarrow$ with probability $1-p$

Let $N_r = \#(\uparrow \text{ steps among the first } r \uparrow, \rightarrow \text{ steps})$. Then

$$\mathbb{P}\left(\overline{\Gamma}_\ell^{(2)} \le n(s+\tfrac{\epsilon}{4})\right) \le \mathbb{P}(N_{\ell-b} \le n(s+\tfrac{\epsilon}{4}) + \epsilon n)$$

(some algebra)

$$\le \mathbb{P}\left(N_{\ell-b} \le (\ell-b)[p-2(p-s)/10]\right)$$
$$\le C' e^{-c'(\ell-b)}$$

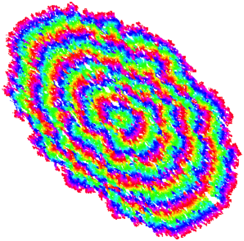
for some C', c' depending only on p, s by Hoeffding's inequality.

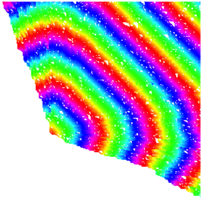
Finally the quadruple sum over $\ell, b, \vec{t}, \vec{x}$ contributes at most $C'' n^3 e^{-c'' n}$.

Combining gives $\mathbb{P}(J_n) \le Ce^{-cn}$ as required.

Open problems

1. Prove Theorem 1 (l.l.n. for the shape) for all $p < p_c(d)$.
2. Is Theorem 2 (c) exhaustive, or are there other directions where $u \in \mathcal{S}_p$?
3. Prove that $n^{-1} T_{nv}(p) \to \zeta_p(v) > 0$ a.s. in a non-trivial subset of directions when $p > p_c(2)$.
4. Prove a shape theorem (in all directions) for the orthant model for $d \geq 2$ when $p \in (1 - p_c(2), p_c(2))$, and/or in a non-trivial subset of directions when $p \notin (1 - p_c(2), p_c(2))$.





Simulation of the set of points reachable in n steps for the orthant model with $p = 1/2$ (left) and $p = 3/4$ (right). White patches surrounded by colour are not reachable from the origin.

Open problems

5. Investigate shape theorems in other degenerate random environments (e.g. arrow e_1 with probability p and full site otherwise).
6. Investigate properties of geodesics (shortest paths) in these models.
7. Generalise to systems where general weights are allowed on directed edges.