Quarter-plane lattice paths with interacting boundaries: the Kreweras and reverse Kreweras models

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Some calculations accompanying the solution to **reverse Kreweras** walks with general boundary weights (a,b,c). Symbols and equation numbers match the manuscript where possible.

Note: Many symbols are reused between this notebook and the Kreweras notebook -- be sure to quit the kernel before switching to the other one, or use a different kernel for each.

(This block needs to be expanded to run some preliminary commands!)

Preliminaries

It will be useful to have some series to substitute into equations to check their validity.

```
In[•]:= (* shorthand to apply a function f to the terms of a series *)
ApplyToSeries[f_, S_] := MapAt[f /@ # &, S, 3]
```

```
In[•]:= (* mathematica sometimes has trouble when
 combining multiple series in the same variable *)
(* so here's a way of dealing with that *)
Simplificate[S_] :=
 Table[S[[1]]^n, {n, S[[-3]] / S[[-1]], S[[-3]] / S[[-1]] + (Length[S[[3]]] - 1) /
       S[[-1]], 1 / S[[-1]]}].S[[3]] + O[S[[1]]]^(S[[-2]] / S[[-1]])
```

```
In[•]:= (* this will also be useful *)
Needs["Notation`"]
```

```
In[•]:= Symbolize[ParsedBoxWrapper[SubscriptBox["_", "_"]]]
Symbolize[ParsedBoxWrapper[SubsuperscriptBox["_", "_", "_"]]]
```

```
In[•]:= (* calculate the coefficients (polynomials in a,b,c) recursively *)
(* let q[n,i,j] be the total weight of
 walks of length n ending at coordinate (i,j) *)
Clear[q]
q[0, 0, 0] = 1;
q[n_, i_, j_] := (q[n, i, j] = 0) /; (n < 0 || i < 0 || j < 0);
q[n_, i_, j_] :=
 (q[n, i, j] = Expand[q[n - 1, i + 1, j + 1] + q[n - 1, i - 1, j] + q[n - 1, i, j - 1]]) /;
  (i > 0 && j > 0)
q[n_, 0, j_] := (q[n, 0, j] = Expand[b q[n - 1, 1, j + 1] + b q[n - 1, 0, j - 1]]) /; (j > 0)
q[n_, i_, 0] := (q[n, i, 0] = Expand[a q[n - 1, i + 1, 1] + a q[n - 1, i - 1, 0]]) /; (i > 0)
q[n_, 0, 0] := (q[n, 0, 0] = Expand[c q[n - 1, 1, 1]])
```

In[•]:=
```
(* then the generating functions *)
Clear[QQ, QQcx, QQcy, QQcxy, QQeval, QQcxeval, QQcyeval, QQdk, QQdkeval]
QQ[N_] := QQ[N] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * x^i * y^j, {n, 0, N}, {i, 0, n}, {j, 0, n}] + O[t] ^ (N + 1)]
(* coefficients of specific powers of x,y, or both *)
QQcx[N_, i_] := QQcx[N, i] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * y^j, {n, 0, N}, {j, 0, n}] + O[t] ^ (N + 1)]
QQcy[N_, j_] := QQcy[N, j] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * x^i, {n, 0, N}, {i, 0, n}] + O[t] ^ (N + 1)]
QQcxy[N_, i_, j_] := QQcxy[N, i, j] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n, {n, 0, N}] + O[t] ^ (N + 1)]
(* evaluating QQ at some other values of (x,y) *)
QQeval[N_, xx_, yy_] := QQeval[N, xx, yy] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * xx^i * yy^j, {n, 0, N}, {i, 0, n}, {j, 0, n}] + O[t] ^ (N + 1)]
QQcxeval[N_, i_, yy_] := QQcxeval[N, i, yy] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * yy^j, {n, 0, N}, {j, 0, n}] + O[t] ^ (N + 1)]
QQcyeval[N_, j_, xx_] := QQcyeval[N, j, xx] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * xx^i, {n, 0, N}, {i, 0, n}] + O[t] ^ (N + 1)]
(* the generalised diagonal term *)
QQdk[N_, k_] := QQdk[N, k] = ApplyToSeries[Expand,
   Sum[q[n, i, i + k] * t^n * x^i, {n, 0, N}, {i, 0, n}] + O[t] ^ (N + 1)]
QQdkeval[N_, k_, xx_] := QQdkeval[N, k, xx] = ApplyToSeries[Expand,
   Sum[q[n, i, i + k] * t^n * xx^i, {n, 0, N}, {i, 0, n}] + O[t] ^ (N + 1)]
```

Section 3

In[•]:=
```
(* the kernel and A,B,G *)
K[x_, y_] := 1 - t (x + y + 1 / x / y)
A = B = G = 1 / x / y
```

Out[•]= $\frac{1}{x\,y}$

In[•]:=
```
(* the rhs of eqn (3.3) *)
mainFE = 1 / c + 1 / a (a - 1 - t a A) Q[x, 0] +
   1 / b (b - 1 - t b B) Q[0, y] + (1 / (a b c) (a c + b c - a b - a b c) + t G) Q[0, 0];
(* then verifying eqn (3.3) *)
mainFE - K[x, y] × Q[x, y] /. {Q[x, y] → QQ[12],
  Q[x, 0] → QQcy[12, 0], Q[0, y] → QQcx[12, 0], Q[0, 0] → QQcxy[12, 0, 0]}
```

Out[•]= $O[t]^{13}$

Section 4.2

In[•]:=
```
(* apply the kernel symmetries *)
mainFE0 = mainFE;
mainFE1 = mainFE0 /. {x → 1 / (x y)};
mainFE2 = mainFE1 /. {y → 1 / (x y)};
mainFE3 = mainFE2 /. {x → 1 / (x y)};
mainFE4 = mainFE3 /. {y → 1 / (x y)};
mainFE5 = mainFE4 /. {x → 1 / (x y)};
```

In[•]:=
```
(* the vector V from eqn (4.5) *)
(* the order is arbitrary *)
V = {Q[x, 0], Q[0, y], Q[1 / x / y, 0], Q[0, 1 / x / y], Q[0, x], Q[y, 0]};
(* then the coefficient matrix M *)
M = {Coefficient[mainFE0, V], Coefficient[mainFE1, V], Coefficient[mainFE2, V],
  Coefficient[mainFE3, V], Coefficient[mainFE4, V], Coefficient[mainFE5, V]}
```

Out[•]=

$$\left\{\left\{\frac{-1+a-\frac{a t}{x y}}{a}, \frac{-1+b-\frac{b t}{x y}}{b}, 0, 0, 0, 0\right\}, \left\{0, \frac{-1+b-b t x}{b}, \frac{-1+a-a t x}{a}, 0, 0, 0\right\}, \left\{0, 0, 0, \frac{-1+b-b t x}{b}, 0, \frac{-1+a-a t x}{a}\right\}, \left\{0, 0, 0, 0, \frac{-1+b-\frac{b t}{x y}}{b}, \frac{-1+a-\frac{a t}{x y}}{a}\right\}, \left\{0, 0, \frac{-1+a-a t y}{a}, 0, \frac{-1+b-b t y}{b}, 0\right\}, \left\{\frac{-1+a-a t y}{a}, 0, 0, \frac{-1+b-b t y}{b}, 0, 0\right\}\right\}$$

In[•]:=
```
(* write this using *)
Ap[x_, y_] := 1 / a (a - 1 - t a / x / y)
Bp[x_, y_] := 1/b (b - 1 - t b/x/y)
{{Ap[x, y], Bp[x, y], 0, 0, 0, 0}, {0, Bp[1 / x / y, y], Ap[1 / x / y, y], 0, 0, 0},
  {0, 0, 0, Bp[y, 1 / x / y], 0, Ap[y, 1 / x / y]},
  {0, 0, 0, 0, Bp[y, x], Ap[y, x]}, {0, 0, Ap[1 / x / y, x], 0, Bp[1 / x / y, x], 0},
  {Ap[x, 1 / x / y], 0, 0, Bp[x, 1 / x / y], 0, 0}} - M
```

Out[•]= {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

In[•]:=
```
(* the vector C is everything else, see eqn (4.7) *)
CC = {mainFE0, mainFE1, mainFE2, mainFE3, mainFE4, mainFE5} /. (# → 0 & /@ V)
```

Out[•]=

$$\left\{\frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+\frac{t}{x y}\right) Q[0, 0], \frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+t x\right) Q[0, 0], \frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+t x\right) Q[0, 0], \frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+\frac{t}{x y}\right) Q[0, 0], \frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+t y\right) Q[0, 0], \frac{1}{c}+\left(\frac{-a b+a c+b c-a b c}{a b c}+t y\right) Q[0, 0]\right\}$$

In[•]:=
```
(* M has rank 5 *)
MatrixRank[M]
```

Out[•]= 5

In[∘]:=
```
(* the vector N spans the nullspace of M, see eqn (4.8) *)
NullSpace[Mᵀ];
(* clean up the denominators a bit *)
NN = -%[[1]] * (-a t - x y + a x y) * (1 - b + b t x) / x // Factor
(* and see *)
NN.M // FullSimplify
```

Out[∘]=
$$\left\{-(1-b+b\,t\,x)\,y\,(1-a+a\,t\,y),\ -\frac{(1-a+a\,t\,y)\,(-b\,t-x\,y+b\,x\,y)}{x},\ \frac{(1-b+b\,t\,y)\,(-a\,t-x\,y+a\,x\,y)}{x},\ (1-a+a\,t\,x)\,y\,(1-b+b\,t\,y),\ \frac{(1-a+a\,t\,x)\,(-b\,t-x\,y+b\,x\,y)}{x},\ -\frac{(1-b+b\,t\,x)\,(-a\,t-x\,y+a\,x\,y)}{x}\right\}$$

Out[∘]= $\{0, 0, 0, 0, 0, 0\}$

In[∘]:=
```
(* we observe that N.C = 0 *)
NN.CC;
FullSimplify[%]
```

Out[∘]= 0

In[∘]:=
```
(* now we need to extract [y^0] of N.Q *)
fullOS =
 NN.{Q[x, y], Q[1 / x / y, y], Q[y, 1 / x / y], Q[y, x], Q[1 / x / y, x], Q[x, 1 / x / y]}
(* check it *)
% /. {Q[ecks_, why_] → QQeval[12, ecks, why]}
```

Out[∘]=
$$-\frac{(1-b+b\,t\,x)\,(-a\,t-x\,y+a\,x\,y)\,Q\left[x,\frac{1}{x\,y}\right]}{x}-(1-b+b\,t\,x)\,y\,(1-a+a\,t\,y)\,Q[x,y]+\frac{(1-a+a\,t\,x)\,(-b\,t-x\,y+b\,x\,y)\,Q\left[\frac{1}{x\,y},x\right]}{x}-\frac{(1-a+a\,t\,y)\,(-b\,t-x\,y+b\,x\,y)\,Q\left[\frac{1}{x\,y},y\right]}{x}+(1-a+a\,t\,x)\,y\,(1-b+b\,t\,y)\,Q[y,x]+\frac{(1-b+b\,t\,y)\,(-a\,t-x\,y+a\,x\,y)\,Q\left[y,\frac{1}{x\,y}\right]}{x}$$

Out[∘]= $O[t]^{13}$

In[•]:=
```
(* this is not too complicated *)
fullOSy0 = {0, 0, 0, 0, 0, 0};
(* the Q[x,y] term *)
CoefficientList[Coefficient[fullOS, Q[x, y]], y]
(* it contributes nothing *)
fullOSy0[[1]] = 0
(* the Q[1/x/y,y] term *)
CoefficientList[Coefficient[fullOS, Q[1 / x / y, y]], y]
(* it contributes some diagonal terms *)
fullOSy0[[2]] = %[[1]] * Q^d_0[1 / x] + %[[2]] * Q^d_-1[1 / x] + %[[3]] * Q^d_-2[1 / x]
(* the Q[y,1/x/y] term *)
CoefficientList[Coefficient[fullOS, Q[y, 1 / x / y]], y]
(* some more diagonal terms *)
fullOSy0[[3]] = %[[1]] * Q^d_0[1 / x] + %[[2]] * Q^d_1[1 / x] / x + %[[3]] * Q^d_2[1 / x] / x^2
(* the Q[y,x] term *)
CoefficientList[Coefficient[fullOS, Q[y, x]], y]
(* gives nothing *)
fullOSy0[[4]] = 0
(* the Q[1/x/y,x] term *)
CoefficientList[Coefficient[fullOS, Q[1 / x / y, x]], y]
(* gives *)
fullOSy0[[5]] = %[[1]] * Q[0, x] + %[[2]] * Q_1,.[x] / x
(* the Q[x,1/x/y] term *)
CoefficientList[Coefficient[fullOS, Q[x, 1 / x / y]], y]
(* gives *)
fullOSy0[[6]] = %[[1]] * Q[x, 0] + %[[2]] * Q_.,1[x] / x
```

Out[•]= $\{0, -1+b-b\,t\,x-a\,(-1+b-b\,t\,x), a\,t\,(-1+b-b\,t\,x)\}$

Out[•]= 0

Out[•]= $\left\{\frac{b\,t}{x}-\frac{a\,b\,t}{x}, 1-a-b+a\,b+\frac{a\,b\,t^2}{x}, a\,t-a\,b\,t\right\}$

Out[•]= $\left(\frac{b\,t}{x}-\frac{a\,b\,t}{x}\right) Q^d_0\left[\frac{1}{x}\right]+\left(1-a-b+a\,b+\frac{a\,b\,t^2}{x}\right) Q^d_{-1}\left[\frac{1}{x}\right]+(a\,t-a\,b\,t)\, Q^d_{-2}\left[\frac{1}{x}\right]$

Out[•]= $\left\{-\frac{a\,t}{x}+\frac{a\,b\,t}{x}, -1+a+b-a\,b-\frac{a\,b\,t^2}{x}, -b\,t+a\,b\,t\right\}$

Out[•]= $\left(-\frac{a\,t}{x}+\frac{a\,b\,t}{x}\right) Q^d_0\left[\frac{1}{x}\right]+\frac{\left(-1+a+b-a\,b-\frac{a\,b\,t^2}{x}\right) Q^d_1\left[\frac{1}{x}\right]}{x}+\frac{(-b\,t+a\,b\,t)\, Q^d_2\left[\frac{1}{x}\right]}{x^2}$

Out[•]= $\{0, 1-a+a\,t\,x-b\,(1-a+a\,t\,x), b\,t\,(1-a+a\,t\,x)\}$

Out[•]= 0

Out[•]= $\left\{-\frac{b\,t\,(1-a+a\,t\,x)}{x}, -1+a-a\,t\,x+b\,(1-a+a\,t\,x)\right\}$

Out[•]= $-\frac{b\,t\,(1-a+a\,t\,x)\,Q[0, x]}{x}+\frac{(-1+a-a\,t\,x+b\,(1-a+a\,t\,x))\,Q_{1,\cdot}[x]}{x}$

Out[•]= $\left\{\frac{a\,t\,(1-b+b\,t\,x)}{x},\ 1-b+b\,t\,x-a\,(1-b+b\,t\,x)\right\}$

Out[•]= $\frac{a\,t\,(1-b+b\,t\,x)\,Q[x,0]}{x}+\frac{(1-b+b\,t\,x-a\,(1-b+b\,t\,x))\,Q_{\cdot,1}[x]}{x}$

In[•]:=
```
Total[fullOSy0] // Collect[#, Q^d_0[1 / x]] &
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x], Q[x, 0] → QQcy[12, 0],
   Q_{1,·}[x] → QQcxeval[12, 1, x], Q_{·,1}[x] → QQcy[12, 1], Q^d_0[1/x] → QQdkeval[12, 0, 1 / x],
   Q^d_{-2}[1/x] → QQdkeval[12, -2, 1 / x], Q^d_{-1}[1/x] → QQdkeval[12, -1, 1 / x],
   Q^d_1[1/x] → QQdkeval[12, 1, 1 / x], Q^d_2[1/x] → QQdkeval[12, 2, 1 / x]}
```

Out[•]= $-\frac{b\,t\,(1-a+a\,t\,x)\,Q[0,x]}{x}+\frac{a\,t\,(1-b+b\,t\,x)\,Q[x,0]}{x}+\frac{(-1+a-a\,t\,x+b\,(1-a+a\,t\,x))\,Q_{1,\cdot}[x]}{x}+\frac{(1-b+b\,t\,x-a\,(1-b+b\,t\,x))\,Q_{\cdot,1}[x]}{x}+\left(-\frac{a\,t}{x}+\frac{b\,t}{x}\right)Q^d_0\left[\frac{1}{x}\right]+\frac{\left(-1+a+b-a\,b-\frac{a\,b\,t^2}{x}\right)Q^d_1\left[\frac{1}{x}\right]}{x}+\frac{(-b\,t+a\,b\,t)\,Q^d_2\left[\frac{1}{x}\right]}{x^2}+\left(1-a-b+a\,b+\frac{a\,b\,t^2}{x}\right)Q^d_{-1}\left[\frac{1}{x}\right]+(a\,t-a\,b\,t)\,Q^d_{-2}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

In[•]:=
```
(* some boundary and diagonal relations
 can be used to eliminate some of these *)
(* the equation for Q[x,0] *)
Qx0eqn = -Q[x, 0] + 1 + t a x Q[x, 0] + t a / x (Q_{·,1}[x] - x Q_{1,1} - Q_{0,1}) + t c Q_{1,1}
(* check it *)
% /. {Q[x, 0] → QQcy[12, 0], Q_{·,1}[x] → QQcy[12, 1],
    Q_{1,1} → QQcxy[12, 1, 1], Q_{0,1} → QQcxy[12, 0, 1]} // Simplify
(* similarly for Q[0,x] *)
Q0xeqn = -Q[0, x] + 1 + t b x Q[0, x] + t b / x (Q_{1,·}[x] - x Q_{1,1} - Q_{1,0}) + t c Q_{1,1}
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x], Q_{1,·}[x] → QQcxeval[12, 1, x],
    Q_{1,1} → QQcxy[12, 1, 1], Q_{1,0} → QQcxy[12, 1, 0]} // Simplify
(* then for diagonals, starting with the -1 *)
Qdm1eqn = -Q^d_{-1}[1 / x] + t x (Q^d_{-1}[1 / x] - Q_{2,1} / x ^ 2 - Q_{1,0} / x) +
   t a / x Q_{2,1} + t / x (Q^d_0[1 / x] - Q[0, 0]) + t a / x Q[0, 0] + t Q^d_{-2}[1 / x]
(* check it *)
% /. {Q^d_{-2}[1 / x] → QQdkeval[12, -2, 1 / x],
   Q^d_{-1}[1 / x] → QQdkeval[12, -1, 1 / x], Q^d_0[1 / x] → QQdkeval[12, 0, 1 / x],
   Q[0, 0] → QQcxy[12, 0, 0], Q_{1,0} → QQcxy[12, 1, 0], Q_{2,1} → QQcxy[12, 2, 1]}
(* then the 0 diagonal *)
Qd0eqn =
 -Q^d_0[1 / x] + 1 + t x (Q^d_0[1 / x] - Q_{1,1} / x - Q[0, 0]) + t c Q_{1,1} + t / x Q^d_1[1 / x] + t Q^d_{-1}[1 / x]
```

```
(* check it *)
% /. {Q^d_0[1 / x] → QQdkeval[12, 0, 1 / x], Q^d_-1[1 / x] → QQdkeval[12, -1, 1 / x],
  Q^d_1[1 / x] → QQdkeval[12, 1, 1 / x], Q[0, 0] → QQcxy[12, 0, 0], Q_{1,1} → QQcxy[12, 1, 1]}
(* then the 1 diagonal *)
Qdp1eqn = -Q^d_1[1 / x] + t x (Q^d_1[1 / x] - Q_{1,2} / x - Q_{0,1}) +
  t b Q_{1,2} + t / x Q^d_2[1 / x] + t (Q^d_0[1 / x] - Q[0, 0]) + t b Q[0, 0]
(* check it *)
% /. {Q^d_2[1 / x] → QQdkeval[12, 2, 1 / x],
  Q^d_1[1 / x] → QQdkeval[12, 1, 1 / x], Q^d_0[1 / x] → QQdkeval[12, 0, 1 / x],
  Q[0, 0] → QQcxy[12, 0, 0], Q_{0,1} → QQcxy[12, 0, 1], Q_{1,2} → QQcxy[12, 1, 2]}
(* now with all these we've introduced some point terms
 that can be eliminated, namely Q_{2,1}, Q_{1,2} and Q_{1,1} *)
Q10eqn = -Q_{1,0} + t a Q[0, 0] + t a Q_{2,1}
(* check it *)
% /. {Q_{1,0} → QQcxy[12, 1, 0], Q[0, 0] → QQcxy[12, 0, 0], Q_{2,1} → QQcxy[12, 2, 1]}
Q01eqn = -Q_{0,1} + t b Q[0, 0] + t b Q_{1,2}
(* check it *)
% /. {Q_{0,1} → QQcxy[12, 0, 1], Q[0, 0] → QQcxy[12, 0, 0], Q_{1,2} → QQcxy[12, 1, 2]}
Q00eqn = -Q[0, 0] + 1 + t c Q_{1,1}
(* check it *)
% /. {Q[0, 0] → QQcxy[12, 0, 0], Q_{1,1} → QQcxy[12, 1, 1]}
(* these ones will be useful in a bit *)
Q11eqn = -Q_{1,1} + t Q_{2,2} + t Q_{0,1} + t Q_{1,0}
(* check it *)
% /. {Q_{1,1} → QQcxy[12, 1, 1], Q_{2,2} → QQcxy[12, 2, 2],
  Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0]}
Q22eqn = -Q_{2,2} + t Q_{3,3} + t Q_{1,2} + t Q_{2,1}
% /. {Q_{2,2} → QQcxy[12, 2, 2], Q_{3,3} → QQcxy[12, 3, 3],
  Q_{1,2} → QQcxy[12, 1, 2], Q_{2,1} → QQcxy[12, 2, 1]}
```

Out[•]= $1 + c\,Q_{1,1}\,t - Q[x,0] + a\,t\,x\,Q[x,0] + \frac{a\,t\left(-Q_{0,1} - Q_{1,1}\,x + Q_{\cdot,1}[x]\right)}{x}$

Out[•]= $O[t]^{13}$

Out[•]= $1 + c\,Q_{1,1}\,t - Q[0,x] + b\,t\,x\,Q[0,x] + \frac{b\,t\left(-Q_{1,0} - Q_{1,1}\,x + Q_{1,\cdot}[x]\right)}{x}$

Out[•]= $O[t]^{13}$

Out[•]= $\frac{a\,Q_{2,1}\,t}{x} + \frac{a\,t\,Q[0,0]}{x} + \frac{t\left(-Q[0,0] + Q^d_0\left[\frac{1}{x}\right]\right)}{x} - Q^d_{-1}\left[\frac{1}{x}\right] + t\,x\left(-\frac{Q_{2,1}}{x^2} - \frac{Q_{1,0}}{x} + Q^d_{-1}\left[\frac{1}{x}\right]\right) + t\,Q^d_{-2}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

Out[•]= $1 + c\,Q_{1,1}\,t - Q^d_0\left[\frac{1}{x}\right] + t\,x\left(-\frac{Q_{1,1}}{x} - Q[0,0] + Q^d_0\left[\frac{1}{x}\right]\right) + \frac{t\,Q^d_1\left[\frac{1}{x}\right]}{x} + t\,Q^d_{-1}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

Out[•]= $b Q_{1,2} t + b t Q[0, 0] + t \left(-Q[0, 0] + Q_0^d\left[\frac{1}{x}\right]\right) - Q_1^d\left[\frac{1}{x}\right] + t x \left(-Q_{0,1} - \frac{Q_{1,2}}{x} + Q_1^d\left[\frac{1}{x}\right]\right) + \frac{t Q_2^d\left[\frac{1}{x}\right]}{x}$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{1,0} + a Q_{2,1} t + a t Q[0, 0]$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{0,1} + b Q_{1,2} t + b t Q[0, 0]$

Out[•]= $O[t]^{13}$

Out[•]= $1 + c Q_{1,1} t - Q[0, 0]$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{1,1} + Q_{0,1} t + Q_{1,0} t + Q_{2,2} t$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{2,2} + Q_{1,2} t + Q_{2,1} t + Q_{3,3} t$

Out[•]= $O[t]^{13}$

In[•]:=
```
(* applying these gives eqn (4.10) *)
fullOSy0 /. Solve[Qx0eqn == 0, Q·,1[x]][[1]];
% /. Solve[Q0xeqn == 0, Q1,·[x]][[1]];
% /. Solve[Qdm1eqn == 0, Q^d_-2[1 / x]][[1]];
% /. Solve[Qdp1eqn == 0, Q^d_2[1 / x]][[1]];
% /. Solve[Qd0eqn == 0, Q^d_-1[1 / x]][[1]];
% /. Solve[Q10eqn == 0, Q2,1][[1]];
% /. Solve[Q01eqn == 0, Q1,2][[1]];
% /. Solve[Q00eqn == 0, Q1,1][[1]];
fullOSy0v2 = Collect[Total[%],
   {Q[0, 0], Q^d_0[1/x], Q^d_1[1/x], Q[0, x], Q[x, 0]}, Collect[#, x, Factor] &]
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x],
   Q[x, 0] → QQcy[12, 0], Q^d_0[1/x] → QQdkeval[12, 0, 1 / x],
   Q^d_1[1/x] → QQdkeval[12, 1, 1 / x], Q[0, 0] → QQcxy[12, 0, 0]}
```

Out[•]=

$$\frac{-1+b}{c\,t}-\frac{a\,b\,t}{c\,x}-\frac{(-2\,a+b+a\,b)\,x}{c}+$$
$$\left(\frac{a\,b-a\,b^2+a\,c-a^2\,c-b\,c-a\,b\,c+a^2\,b\,c+b^2\,c+a^2\,b^2\,c\,t^3}{a\,b\,c\,t}-\frac{a\,b\,(-1+c)\,t}{c\,x}-\right.$$
$$\left.\frac{(2\,a^2\,b-a\,b^2-a^2\,b^2-a^2\,c-a\,b\,c+b^2\,c+a^2\,b^2\,c)\,x}{a\,b\,c}+a\,(-1+b)\,t\,x^2\right)Q[0,0]+$$
$$\left(-\frac{1-a-b+a\,b+a\,b^2\,t^3}{b\,t}+\frac{(-1+a)\,b\,t}{x}+\frac{(-1+b)\,(a-b+a\,b)\,x}{b}-a\,(-1+b)\,t\,x^2\right)Q[0,x]+$$
$$\left(\frac{1-a-b+a\,b+a^2\,b\,t^3}{a\,t}-\frac{a\,(-1+b)\,t}{x}-\frac{(-1+a)\,(-a+b+a\,b)\,x}{a}+(-1+a)\,b\,t\,x^2\right)Q[x,0]+$$
$$\left(-\frac{-1+b+a\,b\,t^3}{t}+\frac{(-2\,a+2\,b+a\,b)\,t}{x}+(1+a)\,(-1+b)\,x-a\,(-1+b)\,t\,x^2\right)Q^d_0\left[\frac{1}{x}\right]+$$
$$\left(-(-a-b+2\,a\,b)\,t-\frac{2\,a\,b\,t^2}{x^2}+\frac{-2+a+b}{x}\right)Q^d_1\left[\frac{1}{x}\right]$$

Out[•]= $O[t]^{12}$

In[•]:=
```
(* we now take the positive and negative parts wrt x *)
(* this is straightforward *)
```

In[•]:=
```
(* for the positive part *)
(*eqn (4.11)*)
fullOSy0xpos = {0, 0, 0, 0, 0, 0};
fullOSy0v2 /. {Q[__] → 0, Q^d_0[1/x] → 0, Q^d_1[1/x] → 0}
fullOSy0xpos[[1]] = Select[%, Exponent[#, x] > 0 &]
Coefficient[fullOSy0v2, Q[0, 0]]
fullOSy0xpos[[2]] = Select[%, Exponent[#, x] > 0 &] * Q[0, 0]
Coefficient[fullOSy0v2, Q[0, x]]
fullOSy0xpos[[3]] = Select[%, Exponent[#, x] > 0 &] * Q[0, x] +
  Select[%, Exponent[#, x] == 0 &] * (Q[0, x] - Q[0, 0]) +
  Select[%, Exponent[#, x] == -1 &] * (Q[0, x] - Q[0, 0] - x Q_{0,1})
Coefficient[fullOSy0v2, Q[x, 0]]
fullOSy0xpos[[4]] = Select[%, Exponent[#, x] > 0 &] * Q[x, 0] +
  Select[%, Exponent[#, x] == 0 &] * (Q[x, 0] - Q[0, 0]) +
  Select[%, Exponent[#, x] == -1 &] * (Q[x, 0] - Q[0, 0] - x Q_{1,0})
Coefficient[fullOSy0v2, Q^d_0[1/x]]
fullOSy0xpos[[5]] = Select[%, Exponent[#, x] == 1 &] * Q[0, 0] +
  Select[%, Exponent[#, x] == 2 &] * (Q[0, 0] + Q_{1,1} / x)
Coefficient[fullOSy0v2, Q^d_1[1/x]]
fullOSy0xpos[[6]] = 0
```

Out[•]=
$$\frac{-1+b}{c\,t}-\frac{a\,b\,t}{c\,x}-\frac{(-2a+b+a\,b)\,x}{c}$$

Out[•]=
$$-\frac{(-2a+b+a\,b)\,x}{c}$$

Out[•]=
$$\frac{a\,b-a\,b^2+a\,c-a^2\,c-b\,c-a\,b\,c+a^2\,b\,c+b^2\,c+a^2\,b^2\,c\,t^3}{a\,b\,c\,t}-\frac{a\,b\,(-1+c)\,t}{c\,x}-\frac{\left(2a^2\,b-a\,b^2-a^2\,b^2-a^2\,c-a\,b\,c+b^2\,c+a^2\,b^2\,c\right)x}{a\,b\,c}+a\,(-1+b)\,t\,x^2$$

Out[•]=
$$\left(-\frac{\left(2a^2\,b-a\,b^2-a^2\,b^2-a^2\,c-a\,b\,c+b^2\,c+a^2\,b^2\,c\right)x}{a\,b\,c}+a\,(-1+b)\,t\,x^2\right)Q[0,0]$$

Out[•]=
$$-\frac{1-a-b+a\,b+a\,b^2\,t^3}{b\,t}+\frac{(-1+a)\,b\,t}{x}+\frac{(-1+b)\,(a-b+a\,b)\,x}{b}-a\,(-1+b)\,t\,x^2$$

Out[•]=
$$\left(\frac{(-1+b)\,(a-b+a\,b)\,x}{b}-a\,(-1+b)\,t\,x^2\right)Q[0,x]-\frac{\left(1-a-b+a\,b+a\,b^2\,t^3\right)\left(-Q[0,0]+Q[0,x]\right)}{b\,t}+\frac{(-1+a)\,b\,t\,\left(-Q_{0,1}\,x-Q[0,0]+Q[0,x]\right)}{x}$$

Out[•]=
$$\frac{1-a-b+a\,b+a^2\,b\,t^3}{a\,t}-\frac{a\,(-1+b)\,t}{x}-\frac{(-1+a)\,(-a+b+a\,b)\,x}{a}+(-1+a)\,b\,t\,x^2$$

Out[•]= $$\left(-\frac{(-1+a)\,(-a+b+a\,b)\,x}{a}+(-1+a)\,b\,t\,x^2\right)Q[x,0]+\frac{\left(1-a-b+a\,b+a^2\,b\,t^3\right)(-Q[0,0]+Q[x,0])}{a\,t}-\frac{a\,(-1+b)\,t\,(-Q_{1,0}\,x-Q[0,0]+Q[x,0])}{x}$$

Out[•]= $$-\frac{-1+b+a\,b\,t^3}{t}+\frac{(-2\,a+2\,b+a\,b)\,t}{x}+(1+a)\,(-1+b)\,x-a\,(-1+b)\,t\,x^2$$

Out[•]= $$(1+a)\,(-1+b)\,x\,Q[0,0]-a\,(-1+b)\,t\,x^2\left(\frac{Q_{1,1}}{x}+Q[0,0]\right)$$

Out[•]= $$-(-a-b+2\,a\,b)\,t-\frac{2\,a\,b\,t^2}{x^2}+\frac{-2+a+b}{x}$$

Out[•]= 0

In[•]:=
```
Total[fullOSy0xpos];
% /. Solve[Q00eqn == 0, Q1,1][[1]];
fullOSy0xposv2 =
 Collect[%, {Q[0, 0], Q[x, 0], Q[0, x], Q0,1, Q1,0}, Collect[#, x, Factor] &]
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x], Q[x, 0] → QQcy[12, 0],
  Q[0, 0] → QQcxy[12, 0, 0], Q0,1 → QQcxy[12, 0, 1], Q1,0 → QQcxy[12, 1, 0]}
```

Out[•]= $$-(-1+a)\,b\,Q_{0,1}\,t+a\,(-1+b)\,Q_{1,0}\,t+\frac{(a-b)\,x}{c}+\left(\frac{(-1+a)\,(a-b)\,(-1+b)}{a\,b\,t}-\frac{(a-b)\,t}{x}-\frac{(a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x}{a\,b\,c}\right)Q[0,0]+\left(-\frac{1-a-b+a\,b+a\,b^2\,t^3}{b\,t}+\frac{(-1+a)\,b\,t}{x}+\frac{(-1+b)\,(a-b+a\,b)\,x}{b}-a\,(-1+b)\,t\,x^2\right)Q[0,x]+\left(\frac{1-a-b+a\,b+a^2\,b\,t^3}{a\,t}-\frac{a\,(-1+b)\,t}{x}-\frac{(-1+a)\,(-a+b+a\,b)\,x}{a}+(-1+a)\,b\,t\,x^2\right)Q[x,0]$$

Out[•]= $O[t]^{12}$

In[•]:=
```
(* then for the negative part wrt x *)
(*eqn (4.12)*)
fullOSy0xneg = {0, 0, 0, 0, 0, 0};
fullOSy0v2 /. {Q[__] → 0, Q0d[1/x] → 0, Q1d[1/x] → 0}
fullOSy0xneg[[1]] = Select[%, Exponent[#, x] < 0 &]
Coefficient[fullOSy0v2, Q[0, 0]]
fullOSy0xneg[[2]] = Select[%, Exponent[#, x] < 0 &] * Q[0, 0]
Coefficient[fullOSy0v2, Q[0, x]]
fullOSy0xneg[[3]] = Select[%, Exponent[#, x] == -1 &] * Q[0, 0]
Coefficient[fullOSy0v2, Q[x, 0]]
fullOSy0xneg[[4]] = Select[%, Exponent[#, x] == -1 &] * Q[0, 0]
Coefficient[fullOSy0v2, Q0d[1/x]]
fullOSy0xneg[[5]] = Select[%, Exponent[#, x] < 0 &] * Q0d[1/x] +
  Select[%, Exponent[#, x] == 0 &] * (Q0d[1/x] - Q[0, 0]) +
  Select[%, Exponent[#, x] == 1 &] * (Q0d[1/x] - Q[0, 0] - Q1,1 / x) +
  Select[%, Exponent[#, x] == 2 &] * (Q0d[1/x] - Q[0, 0] - Q1,1 / x - Q2,2 / x^2)
Coefficient[fullOSy0v2, Q1d[1/x]]
fullOSy0xneg[[6]] = Select[%, Exponent[#, x] < 0 &] * Q1d[1/x] +
  Select[%, Exponent[#, x] == 0 &] * (Q1d[1/x] - Q0,1)
```

Out[•]= $\frac{-1+b}{c\,t} - \frac{a\,b\,t}{c\,x} - \frac{(-2\,a+b+a\,b)\,x}{c}$

Out[•]= $-\frac{a\,b\,t}{c\,x}$

Out[•]= $\frac{a\,b - a\,b^2 + a\,c - a^2\,c - b\,c - a\,b\,c + a^2\,b\,c + b^2\,c + a^2\,b^2\,c\,t^3}{a\,b\,c\,t} - \frac{a\,b\,(-1+c)\,t}{c\,x} - \frac{(2\,a^2\,b - a\,b^2 - a^2\,b^2 - a^2\,c - a\,b\,c + b^2\,c + a^2\,b^2\,c)\,x}{a\,b\,c} + a\,(-1+b)\,t\,x^2$

Out[•]= $-\frac{a\,b\,(-1+c)\,t\,Q[0,0]}{c\,x}$

Out[•]= $-\frac{1 - a - b + a\,b + a\,b^2\,t^3}{b\,t} + \frac{(-1+a)\,b\,t}{x} + \frac{(-1+b)\,(a-b+a\,b)\,x}{b} - a\,(-1+b)\,t\,x^2$

Out[•]= $\frac{(-1+a)\,b\,t\,Q[0,0]}{x}$

Out[•]= $\frac{1 - a - b + a\,b + a^2\,b\,t^3}{a\,t} - \frac{a\,(-1+b)\,t}{x} - \frac{(-1+a)\,(-a+b+a\,b)\,x}{a} + (-1+a)\,b\,t\,x^2$

Out[•]= $-\frac{a\,(-1+b)\,t\,Q[0,0]}{x}$

Out[∘]= $-\frac{-1+b+a\,b\,t^3}{t}+\frac{(-2\,a+2\,b+a\,b)\,t}{x}+(1+a)\,(-1+b)\,x-a\,(-1+b)\,t\,x^2$

Out[∘]= $\frac{(-2\,a+2\,b+a\,b)\,t\,Q^d_0\left[\frac{1}{x}\right]}{x}-\frac{(-1+b+a\,b\,t^3)\,\left(-Q[0,\,0]+Q^d_0\left[\frac{1}{x}\right]\right)}{t}+$
$(1+a)\,(-1+b)\,x\,\left(-\frac{Q_{1,1}}{x}-Q[0,\,0]+Q^d_0\left[\frac{1}{x}\right]\right)-$
$a\,(-1+b)\,t\,x^2\,\left(-\frac{Q_{2,2}}{x^2}-\frac{Q_{1,1}}{x}-Q[0,\,0]+Q^d_0\left[\frac{1}{x}\right]\right)$

Out[∘]= $-(-a-b+2\,a\,b)\,t-\frac{2\,a\,b\,t^2}{x^2}+\frac{-2+a+b}{x}$

Out[∘]= $\left(-\frac{2\,a\,b\,t^2}{x^2}+\frac{-2+a+b}{x}\right)\,Q^d_1\left[\frac{1}{x}\right]-(-a-b+2\,a\,b)\,t\,\left(-Q_{0,1}+Q^d_1\left[\frac{1}{x}\right]\right)$

In[∘]:=
```
Total[fullOSy0xneg];
% /. Solve[Q11eqn == 0, Q2,2][[1]];
% /. Solve[Q00eqn == 0, Q1,1][[1]];
fullOSy0xnegv2 =
 Collect[%, {Q[0, 0], Qd0[1/x], Qd1[1/x], Q0,1, Q1,0}, Collect[#, x, Factor] &]
(* check it *)
% /. {Q[0, 0] → QQcxy[12, 0, 0], Q0,1 → QQcxy[12, 0, 1], Q1,0 → QQcxy[12, 1, 0],
  Qd0[1/x] → QQdkeval[12, 0, 1 / x], Qd1[1/x] → QQdkeval[12, 1, 1 / x]}
```

Out[∘]= $\frac{-1+b}{c\,t}+(-1+a)\,b\,Q_{0,1}\,t-a\,(-1+b)\,Q_{1,0}\,t-\frac{a\,b\,t}{c\,x}-$
$\frac{a\,(-1+b)\,x}{c}+\left(\frac{1-b-c+b\,c+a\,b\,c\,t^3}{c\,t}-\frac{(-a\,b-a\,c+b\,c+a\,b\,c)\,t}{c\,x}-\right.$
$\left.\frac{(-1+b)\,(-a+c+a\,c)\,x}{c}+a\,(-1+b)\,t\,x^2\right)\,Q[0,\,0]+$
$\left(-\frac{-1+b+a\,b\,t^3}{t}+\frac{(-2\,a+2\,b+a\,b)\,t}{x}+(1+a)\,(-1+b)\,x-a\,(-1+b)\,t\,x^2\right)\,Q^d_0\left[\frac{1}{x}\right]+$
$\left(-(-a-b+2\,a\,b)\,t-\frac{2\,a\,b\,t^2}{x^2}+\frac{-2+a+b}{x}\right)\,Q^d_1\left[\frac{1}{x}\right]$

Out[∘]= $O[t]^{12}$

Section 4.3

In[∘]:=
```
(* we now compute the half-orbit sum *)
```

In[∘]:=
```
(* the vector V2 from eqn (4.13) *)
V2 = {Q[0, y], Q[1 / x / y, 0], Q[0, 1 / x / y], Q[y, 0]};
(* then the coefficient matrix M2 *)
M2 = {Coefficient[mainFE0, V2], Coefficient[mainFE1, V2],
  Coefficient[mainFE2, V2], Coefficient[mainFE3, V2],
  Coefficient[mainFE4, V2], Coefficient[mainFE5, V2]}
```

Out[∘]=
$$\left\{\left\{\frac{-1+b-\frac{b\,t}{x\,y}}{b},0,0,0\right\},\left\{\frac{-1+b-b\,t\,x}{b},\frac{-1+a-a\,t\,x}{a},0,0\right\},\left\{0,0,\frac{-1+b-b\,t\,x}{b},\frac{-1+a-a\,t\,x}{a}\right\},\left\{0,0,0,\frac{-1+a-\frac{a\,t}{x\,y}}{a}\right\},\left\{0,\frac{-1+a-a\,t\,y}{a},0,0\right\},\left\{0,0,\frac{-1+b-b\,t\,y}{b},0\right\}\right\}$$

In[∘]:=
```
(* the vector C2 is everything else, see eqn (4.13) *)
CC2 = {mainFE0, mainFE1, mainFE2, mainFE3, mainFE4, mainFE5} /.
  {Q[0, y] → 0, Q[1 / x / y, 0] → 0, Q[0, 1 / x / y] → 0, Q[y, 0] → 0}
```

Out[∘]=
$$\left\{\frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+\frac{t}{x\,y}\right)Q[0,0]+\frac{\left(-1+a-\frac{a\,t}{x\,y}\right)Q[x,0]}{a},\right.$$
$$\frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+t\,x\right)Q[0,0],\ \frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+t\,x\right)Q[0,0],$$
$$\frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+\frac{t}{x\,y}\right)Q[0,0]+\frac{\left(-1+b-\frac{b\,t}{x\,y}\right)Q[0,x]}{b},$$
$$\frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+t\,y\right)Q[0,0]+\frac{(-1+b-b\,t\,y)\,Q[0,x]}{b},$$
$$\left.\frac{1}{c}+\left(\frac{-a\,b+a\,c+b\,c-a\,b\,c}{a\,b\,c}+t\,y\right)Q[0,0]+\frac{(-1+a-a\,t\,y)\,Q[x,0]}{a}\right\}$$

In[∘]:=
```
(* M₂ has rank 4 *)
MatrixRank[M₂]
(* so we have two choices for the nullspace vector N₂ *)
NullSpace[(M₂)ᵀ]
(* choose this one, see eqn (4.15) *)
NN₂ = Select[%, Last[#] == 0 &][[1]] * (1 - a + a t x) (-b t - x y + b x y) / y // Factor
(* check *)
NN₂.M₂ // Simplify
```

Out[∘]= 4

Out[∘]= $\left\{\left\{0, 0, -\frac{1-b+b t y}{1-b+b t x}, -\frac{x(1-a+a t x) y(1-b+b t y)}{(1-b+b t x)(-a t-x y+a x y)}, 0, 1\right\}, \left\{-\frac{x(1-b+b t x) y(1-a+a t y)}{(1-a+a t x)(-b t-x y+b x y)}, -\frac{1-a+a t y}{1-a+a t x}, 0, 0, 1, 0\right\}\right\}$

Out[∘]= $\left\{-x(1-b+b t x)(1-a+a t y), -\frac{(1-a+a t y)(-b t-x y+b x y)}{y}, 0, 0, -\frac{(1-a+a t x)(b t+x y-b x y)}{y}, 0\right\}$

Out[∘]= {0, 0, 0, 0}

In[∘]:=
```
(* this time we divide by the kernel and take the y^0 term,
as per eqn (4.16) *)
```

In[∘]:=
```
(* the LHS is straightforward *)
halfOSlhs =
 NN₂.{Q[x, y], Q[1 / x / y, y], Q[y, 1 / x / y], Q[y, x], Q[1 / x / y, x], Q[x, 1 / x / y]}
```

Out[∘]= $-x(1-b+b t x)(1-a+a t y) Q[x, y] - \frac{(1-a+a t x)(b t+x y-b x y) Q\left[\frac{1}{x y}, x\right]}{y} - \frac{(1-a+a t y)(-b t-x y+b x y) Q\left[\frac{1}{x y}, y\right]}{y}$

```
In[•]:= (* eqn (4.17)*)
halfOSlhsy0 = {0, 0, 0};
Coefficient[halfOSlhs, Q[x, y]] // Collect[#, y] &
halfOSlhsy0[[1]] = Coefficient[%, y, 0] * Q[x, 0]
Coefficient[halfOSlhs, Q[1 / x / y, y]] // Collect[#, y] &
halfOSlhsy0[[2]] = Coefficient[%, y, -1] * Q^d_1[1 / x] +
  Coefficient[%, y, 0] * Q^d_0[1 / x] + Coefficient[%, y, 1] * Q^d_-1[1 / x]
Coefficient[halfOSlhs, Q[1 / x / y, x]] // Collect[#, y] &
halfOSlhsy0[[3]] = Coefficient[%, y, 0] * Q[0, x]
```

Out[•]= $-(1-a)\,x\,\left(1-b+b\,t\,x\right)-a\,t\,x\,\left(1-b+b\,t\,x\right)\,y$

Out[•]= $-(1-a)\,x\,\left(1-b+b\,t\,x\right)\,Q[x,0]$

Out[•]= $a\,b\,t^2+x-a\,x-b\,x+a\,b\,x-\dfrac{-b\,t+a\,b\,t}{y}-\left(-a\,t\,x+a\,b\,t\,x\right)\,y$

Out[•]= $\left(a\,b\,t^2+x-a\,x-b\,x+a\,b\,x\right)\,Q^d_0\left[\dfrac{1}{x}\right]+\left(b\,t-a\,b\,t\right)\,Q^d_1\left[\dfrac{1}{x}\right]+\left(a\,t\,x-a\,b\,t\,x\right)\,Q^d_{-1}\left[\dfrac{1}{x}\right]$

Out[•]= $-\left(x-b\,x\right)\,(1-a+a\,t\,x)-\dfrac{b\,t\,(1-a+a\,t\,x)}{y}$

Out[•]= $-\left(x-b\,x\right)\,(1-a+a\,t\,x)\,Q[0,x]$

```
In[•]:= halfOSlhsy0;
% /. Solve[Qd0eqn == 0, Q^d_-1[1 / x]] [[1]];
% /. Solve[Q00eqn == 0, Q_1,1] [[1]];
halfOSlhsy0v2 = Collect[Total[%],
  {Q[0, 0], Q^d_0[1/x], Q^d_1[1/x], Q[0, x], Q[x, 0]}, Collect[#, x, Factor] &]
```

Out[•]= $$\frac{a\,\left(-1+b\right)\,x}{c}+\left(\frac{a\,\left(-1+b\right)\,(-1+c)\,x}{c}-a\,\left(-1+b\right)\,t\,x^2\right)\,Q[0,0]+$$
$$\left(-(-1+a)\,\left(-1+b\right)\,x+a\,\left(-1+b\right)\,t\,x^2\right)\,Q[0,x]+$$
$$\left(-(-1+a)\,\left(-1+b\right)\,x+(-1+a)\,b\,t\,x^2\right)\,Q[x,0]+$$
$$\left(a\,b\,t^2+\left(1-b\right)\,x+a\,\left(-1+b\right)\,t\,x^2\right)\,Q^d_0\left[\frac{1}{x}\right]-\left(a-b\right)\,t\,Q^d_1\left[\frac{1}{x}\right]$$

```
In[•]:= (* can check this manually *)
halfOSlhs;
% /. Q[ecks_, why_] → QQeval[10, ecks, why];
ApplyToSeries[Expand@*Simplify, %];
ApplyToSeries[Coefficient[# + y^π + y^(2 π), y, 0] &, %];
halfOSlhsy0v2 /.
  {Q[0, 0] → QQcxy[10, 0, 0], Q[x, 0] → QQcy[10, 0], Q[0, x] → QQcxeval[10, 0, x],
   Q^d_0[1/x] → QQdkeval[10, 0, 1 / x], Q^d_1[1/x] → QQdkeval[10, 1, 1 / x]};
ApplyToSeries[Expand@*Simplify, %];
% - %%% // Simplify
```

Out[•]= $O[t]^{11}$

In[•]:=
```
(* now for the RHS *)
(* this will be divided by the kernel *)
halfOSrhs = NN₂.CC₂ // Collect[#, Q[__], Collect[#, y, Factor] &] &
```

Out[•]=

$$\frac{a\,b\,t^2 - x + a\,x + b\,x - a\,b\,x - a\,t\,x^2 - b\,t\,x^2 + 2\,a\,b\,t\,x^2}{c} - \frac{a\,b\,t^2\,x}{c\,y} - \frac{a\,b\,t^2\,x^2\,y}{c} + \left(-\frac{1}{a\,b\,c}\left(a^2\,b^2\,t^2 - a^2\,b^2\,c\,t^2 - a\,b\,x + a^2\,b\,x + a\,b^2\,x - a^2\,b^2\,x + a\,c\,x - a^2\,c\,x + b\,c\,x - 3\,a\,b\,c\,x + 2\,a^2\,b\,c\,x - b^2\,c\,x + 2\,a\,b^2\,c\,x - a^2\,b^2\,c\,x + a^2\,b^2\,c\,t^3\,x - a^2\,b\,t\,x^2 - a\,b^2\,t\,x^2 + 2\,a^2\,b^2\,t\,x^2 + a^2\,c\,t\,x^2 + a\,b\,c\,t\,x^2 - 2\,a^2\,b\,c\,t\,x^2 + b^2\,c\,t\,x^2 - 2\,a\,b^2\,c\,t\,x^2 + a^2\,b^2\,c\,t\,x^2\right) + \frac{t\,(-c + a\,c + b\,c - a\,b\,c + a\,b\,t\,x - a\,c\,t\,x - b\,c\,t\,x + a\,b\,c\,t\,x)}{c\,y} + \frac{t\,x\,(-c + a\,c + b\,c - a\,b\,c + a\,b\,t\,x - a\,c\,t\,x - b\,c\,t\,x + a\,b\,c\,t\,x)\,y}{c}\right) Q[0,0] + \left(\frac{(b^2\,t^2 + x - 2\,b\,x + b^2\,x)\,(1 - a + a\,t\,x)}{b} - \frac{(-1+b)\,t\,(1 - a + a\,t\,x)}{y} - (-1+b)\,t\,x\,(1 - a + a\,t\,x)\,y\right) Q[0,x] + \left(\frac{(a^2\,t^2 + x - 2\,a\,x + a^2\,x)\,(1 - b + b\,t\,x)}{a} - \frac{(-1+a)\,t\,(1 - b + b\,t\,x)}{y} - (-1+a)\,t\,x\,(1 - b + b\,t\,x)\,y\right) Q[x,0]$$

Section 4.4

In[•]:=
```
(* this requires factoring the kernel as per eqn (4.21) *)
(* the roots of K *)
Δ = (1 - t x) ^2 - 4 t^2 / x;
Y₀ = (1 - t x - Sqrt[Δ]) / (2 t);
Y₁ = (1 - t x + Sqrt[Δ]) / (2 t);
{K[x, Y₀], K[x, Y₁]} // FullSimplify
ApplyToSeries[Expand@*PowerExpand, Series[Y₀, {t, 0, 3}]]
ApplyToSeries[Expand@*PowerExpand, Series[Y₁, {t, 0, 3}]]
```

Out[•]= $\{0, 0\}$

Out[•]= $\frac{t}{x} + t^2 + \left(\frac{1}{x^2} + x\right) t^3 + O[t]^4$

Out[•]= $\frac{1}{t} - x - \frac{t}{x} - t^2 + \left(-\frac{1}{x^2} - x\right) t^3 + O[t]^4$

In[•]:=
```
(* and then eqn (4.21) *)
1 / K[x, y] - 1 / Sqrt[Δ] (1 / (1 - Y₀ / y) + 1 / (1 - y / Y₁) - 1) // Simplify
```

Out[•]= 0

In[∘]:=
```
(* so now we can compute the y^0 term of the RHS *)
Coefficient[half0Srhs, y, -1] / Y1 / Sqrt[Δ] +
  Coefficient[half0Srhs, y, 0] / Sqrt[Δ] +
  Coefficient[half0Srhs, y, 1] * Y0 / Sqrt[Δ];
half0Srhsy0 = Collect[%, Q[__], Simplify]
```

Out[∘]=

$$\Bigg(\Bigg(-x\left(1+b\left(-1+t x\right)\right)\left(1-t x+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+ a\left(x\left(-1+t x\right)\left(-1+t x-\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+ b\left(-5 t^3 x-x\left(1+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+t^2\left(1-2 x^3+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+t x^2\left(3+2\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)\right)\right)\Bigg)\Bigg/ \left(c\sqrt{-\frac{4 t^2}{x}+\left(-1+t x\right)^2}\left(1-t x+\sqrt{-\frac{4 t^2}{x}+\left(-1+t x\right)^2}\right)\right)+$$

$$\Bigg(\Bigg(-b c x\left(1+b\left(-1+t x\right)\right)\left(1-t x+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)-a\left(1+b\left(-1+t x\right)\right) \left(b x\left(-1+t x-\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+c x\left(1-t x+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+ b c\left(4 t^2+2 t x^2-2 x\left(1+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)\right)\right)+ a^2\left(-c x\left(-1+t x\right)\left(1-t x+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)- b\left(-1+t x\right)\left(x\left(-1+t x-\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+ c\left(4 t^2+2 t x^2-2 x\left(1+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)\right)\right)+ b^2\left(c t^4 x^2+\left(1+c\right) x\left(1+\sqrt{1-2 t x+\frac{t^2\left(-4+x^3\right)}{x}}\right)+\right.\right.$$

$$t^3 \left(5 x + 2 c x - c x \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right) -$$

$$t x^2 \left(3 + 2 \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + c \left(2 + \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right)\right) +$$

$$t^2 \left(-1 + 2 x^3 - \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + c\right.$$

$$\left.\left(-3 + x^3 + \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right)\right)\Bigg)\Bigg)\Bigg)\Bigg) \, Q[0, 0]\Bigg) \Big/$$

$$\left(a b c \sqrt{-\frac{4 t^2}{x} + (-1 + t x)^2} \left(1 - t x + \sqrt{-\frac{4 t^2}{x} + (-1 + t x)^2}\right)\right) +$$

$$\Bigg((1 + a (-1 + t x))$$

$$\left(x - t x^2 + x \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + \right.$$

$$b^2 \left(x - t^3 x - t x^2 + x \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + t^2 \left(-3 + \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right)\right) +$$

$$\left.b \left(4 t^2 + 2 t x^2 - 2 x \left(1 + \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right)\right)\right) Q[0, x]\Bigg) \Big/$$

$$\left(b \sqrt{-\frac{4 t^2}{x} + (-1 + t x)^2} \left(1 - t x + \sqrt{-\frac{4 t^2}{x} + (-1 + t x)^2}\right)\right) +$$

$$\Bigg(\left(1 + b (-1 + t x)\right)$$

$$\left(x - t x^2 + x \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + \right.$$

$$a^2 \left(x - t^3 x - t x^2 + x \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}} + t^2 \left(-3 + \sqrt{1 - 2 t x + \frac{t^2 \left(-4 + x^3\right)}{x}}\right)\right) +$$

$$a\left(4t^2+2tx^2-2x\left(1+\sqrt{1-2tx+\frac{t^2\left(-4+x^3\right)}{x}}\right)\right)Q[x,0]\Big/$$
$$\left(a\sqrt{-\frac{4t^2}{x}+(-1+tx)^2}\left(1-tx+\sqrt{-\frac{4t^2}{x}+(-1+tx)^2}\right)\right)$$

In[∘]:=
```
(* we can check it *)
halfOSrhs / K[x, y] /. Q[ecks_, why_] → QQeval[10, ecks, why];
ApplyToSeries[Expand, %];
ApplyToSeries[Select[# + y^π + y^(2π), Exponent[#, y] == 0 &] &, %];
halfOSrhsy0 /. Q[ecks_, why_] → QQeval[10, ecks, why];
ApplyToSeries[Expand, %];
% - %%%
```

Out[∘]= $O[t]^{11}$

In[∘]:=
```
(* and check it some more *)
halfOSlhsy0v2 - halfOSrhsy0;
% /. {Q[ecks_, why_] → QQeval[10, ecks, why],
   Q0d[1/x] → QQdkeval[10, 0, 1/x], Q1d[1/x] → QQdkeval[10, 1, 1/x]}
```

Out[∘]= $O[t]^{11}$

In[∘]:=
```
(* now we can use the equations from the
 full orbit sum to eliminate Q[0,x] and Q1d[1/x] *)
(* and get to eqn (4.26) *)
halfOSy0 =
  halfOSlhsy0v2 - halfOSrhsy0 /. Solve[fullOSy0xposv2 == 0, Q[0, x]][[1]] /.
   Solve[fullOSy0xnegv2 == 0, Q1d[1/x]][[1]];
(* clean up some denominators *)
halfOSy0v2 = halfOSy0 * a c √Δ * (a x (1 + t x) + x (-2 + b + b t x) - 2 a b t (t + x^2));
μx,0 = Simplify[-Coefficient[halfOSy0v2, Q[x, 0]]] // Factor
ν0d = -Coefficient[halfOSy0v2, Q0d[1/x]] / Sqrt[Δ] // Simplify // Factor
Coefficient[halfOSy0v2, Q0,1] // Simplify;
(Numerator[%] * (-1 + t x + √Δ) // Expand // Simplify) /
  (Denominator[%] * (-1 + t x + √Δ) // Expand // Simplify);
μ0,1 = % /. Sqrt[1 - 2 t x + t^2 (-4 + x^3)/x] → 0 // Factor
ν0,1 = Coefficient[%%, Sqrt[1 - 2 t x + t^2 (-4 + x^3)/x]] // Factor
Coefficient[halfOSy0v2, Q1,0] // Simplify;
(Numerator[%] * (-1 + t x + √Δ) // Expand // Simplify) /
```

$\left(\text{Denominator}[\%] * \left(-1 + t\,x + \sqrt{\Delta}\right) \text{ // Expand // Simplify}\right);$

$\mu_{1,0} = \% \text{ /. } \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}} \to 0 \text{ // Factor}$

$\nu_{1,0} = \text{Coefficient}\left[\%\%, \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}}\right] \text{ // Factor}$

```
Coefficient[half0Sy0v2, Q[0, 0]] // Simplify;
(Numerator[%] * (1 - t x - Sqrt[Δ]) // Expand // Simplify) /
  (Denominator[%] * (1 - t x - Sqrt[Δ]) // Expand // Simplify);
```

$\mu_{0,0} = \% \text{ /. } \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}} \to 0 \text{ // Factor}$

$\nu_{0,0} = \text{Coefficient}\left[\%\%, \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}}\right] \text{ // Factor}$

$\text{half0Sy0v2 /. } \left\{Q[\_\_] \to 0,\ Q_{1,0} \to 0,\ Q_{0,1} \to 0,\ Q_0^d\left[\frac{1}{x}\right] \to 0\right\} \text{ // Simplify // Factor;}$

```
(Numerator[%] * (1 - t x - Sqrt[Δ]) // Expand // Simplify) /
  (Denominator[%] * (1 - t x - Sqrt[Δ]) // Expand // Simplify);
```

$\mu = \% \text{ /. } \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}} \to 0 \text{ // Factor}$

$\nu = \text{Coefficient}\left[\%\%, \sqrt{1 - 2\,t\,x + \frac{t^2\,(-4 + x^3)}{x}}\right] \text{ // Factor}$

Out[•]= $-2\,c\,(1 - b + b\,t\,x)\,(a^2\,t^2 + x - a\,x - a\,t\,x^2 + a^2\,t\,x^2)\,(2\,a\,b\,t^2 + 2\,x - a\,x - b\,x - a\,t\,x^2 - b\,t\,x^2 + 2\,a\,b\,t\,x^2)$

Out[•]= $2\,a\,c\,(a^2\,t^2 + x - a\,x - a\,t\,x^2 + a^2\,t\,x^2)\,(b^2\,t^2 + x - b\,x - b\,t\,x^2 + b^2\,t\,x^2)$

Out[•]= $-(-1 + a)\,a\,b\,c\,t^2\,x\,(2\,a\,b\,t^2 + 2\,x - a\,x - b\,x - a\,t\,x^2 - b\,t\,x^2 + 2\,a\,b\,t\,x^2)$

Out[•]= $(-1 + a)\,a\,(a - b)\,b\,c\,t^2\,x^2$

Out[•]= $a^2\,(-1 + b)\,c\,t^2\,x\,(2\,a\,b\,t^2 + 2\,x - a\,x - b\,x - a\,t\,x^2 - b\,t\,x^2 + 2\,a\,b\,t\,x^2)$

Out[•]= $-a^2\,(a - b)\,(-1 + b)\,c\,t^2\,x^2$

Out[•]= $-(2\,a\,b\,t^2 + 2\,x - a\,x - b\,x - a\,t\,x^2 - b\,t\,x^2 + 2\,a\,b\,t\,x^2)\,(a^2\,b\,t^2 + a^2\,c\,t^2 - a\,b\,c\,t^2 - a^2\,b\,c\,t^2 - a\,x + a^2\,x + a\,b\,x - a^2\,b\,x + 2\,c\,x - 2\,a\,c\,x - 2\,b\,c\,x + 2\,a\,b\,c\,x + a^2\,b\,c\,t^3\,x - 2\,a\,b\,t\,x^2 + a^2\,b\,t\,x^2 + a^2\,c\,t\,x^2 + 2\,b\,c\,t\,x^2 - a\,b\,c\,t\,x^2 - a^2\,b\,c\,t\,x^2 + a^2\,b\,t^2\,x^3 - a^2\,c\,t^2\,x^3 - a\,b\,c\,t^2\,x^3 + a^2\,b\,c\,t^2\,x^3)$

Out[•]= $-a\,x\,(a^2\,b\,t^2 + a\,b^2\,t^2 - 2\,a^2\,b^2\,t^2 - a^2\,c\,t^2 + a^2\,b\,c\,t^2 - b^2\,c\,t^2 + a\,b^2\,c\,t^2 + a\,x - a^2\,x + b\,x - 2\,a\,b\,x + a^2\,b\,x - b^2\,x + a\,b^2\,x - 2\,c\,x + 2\,a\,c\,x + 2\,b\,c\,x - 2\,a\,b\,c\,x + 2\,a^2\,b^2\,t^3\,x - a^2\,b\,c\,t^3\,x - a\,b^2\,c\,t^3\,x + a^2\,b\,t\,x^2 + a\,b^2\,t\,x^2 - 2\,a^2\,b^2\,t\,x^2 - a^2\,c\,t\,x^2 + a^2\,b\,c\,t\,x^2 - b^2\,c\,t\,x^2 + a\,b^2\,c\,t\,x^2 - a^2\,b\,t^2\,x^3 - a\,b^2\,t^2\,x^3 + 2\,a^2\,b^2\,t^2\,x^3 + a^2\,c\,t^2\,x^3 - a^2\,b\,c\,t^2\,x^3 + b^2\,c\,t^2\,x^3 - a\,b^2\,c\,t^2\,x^3)$

Out[•]= $a\left(2\,a\,b\,t^2+2\,x-a\,x-b\,x-a\,t\,x^2-b\,t\,x^2+2\,a\,b\,t\,x^2\right)\left(a\,b\,t^2-x+a\,x+b\,x-a\,b\,x-2\,b\,t\,x^2+a\,b\,t\,x^2+a\,b\,t^2\,x^3\right)$

Out[•]= $a\,x\left(a^2\,b\,t^2+a\,b^2\,t^2-2\,a^2\,b^2\,t^2+a\,x-a^2\,x+b\,x-2\,a\,b\,x+a^2\,b\,x-b^2\,x+a\,b^2\,x+2\,a^2\,b^2\,t^3\,x+a^2\,b\,t\,x^2+a\,b^2\,t\,x^2-2\,a^2\,b^2\,t\,x^2-a^2\,b\,t^2\,x^3-a\,b^2\,t^2\,x^3+2\,a^2\,b^2\,t^2\,x^3\right)$

In[•]:=
```
(* then check all that *)
-μ_{x,0} Q[x, 0] - ν_0^d Sqrt[Δ] Q_0^d[1/x] + (μ + ν Sqrt[Δ]) +
  (μ_{0,0} + ν_{0,0} Sqrt[Δ]) Q[0, 0] + (μ_{0,1} + ν_{0,1} Sqrt[Δ]) Q_{0,1} + (μ_{1,0} + ν_{1,0} Sqrt[Δ]) Q_{1,0} /.
 {Q[x, 0] → QQcy[12, 0], Q_0^d[1/x] → QQdkeval[12, 0, 1 / x],
  Q[0, 0] → QQcxy[12, 0, 0], Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0]}
```

Out[•]= $O[t]^{13}$

Section 4.5

In[•]:=
```
(* the factorisation of Δ *)
(* using Root instead of radicals seems to improve performance *)
(* different versions of Mathematica may take Root[...] in different orders,
so let's not make any assumptions *)
Off[Root::sbr]
d1 = Root[-4 t^2 + # - 2 t #^2 + t^2 #^3 &, 1];
d2 = Root[-4 t^2 + # - 2 t #^2 + t^2 #^3 &, 2];
d3 = Root[-4 t^2 + # - 2 t #^2 + t^2 #^3 &, 3];
X1 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 1}]] == 0 &][[1]]
X2 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 1}]] == 1 / t + 2 Sqrt[t] &][[1]]
X3 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 1}]] == 1 / t - 2 Sqrt[t] &][[1]]
(* then eqns (4.33)-(4.35) *)
Series[{X1, X2, X3}, {t, 0, 10}]
```

Out[•]= Root[-4 t^2 + #1 - 2 t #1^2 + t^2 #1^3 &, 1]

Out[•]= Root[-4 t^2 + #1 - 2 t #1^2 + t^2 #1^3 &, 3]

Out[•]= Root[-4 t^2 + #1 - 2 t #1^2 + t^2 #1^3 &, 2]

Out[•]= $\left\{4\,t^2+32\,t^5+448\,t^8+O[t]^{11},\ \frac{1}{t}+2\sqrt{t}-2\,t^2+5\,t^{7/2}-16\,t^5+\frac{231\,t^{13/2}}{4}-224\,t^8+\frac{7293\,t^{19/2}}{8}+O[t]^{21/2},\ \frac{1}{t}-2\sqrt{t}-2\,t^2-5\,t^{7/2}-16\,t^5-\frac{231\,t^{13/2}}{4}-224\,t^8-\frac{7293\,t^{19/2}}{8}+O[t]^{21/2}\right\}$

In[•]:=
```
(* then the factorisation *)
Δ0 = t^2 X2 X3;
Δp = (1 - x / X2) (1 - x / X3);
Δm = 1 - X1 / x;
(* so that *)
Δ - Δ0 Δp Δm // FullSimplify
```

Out[•]= 0

In[∘]:=
```
(* and then verifying eqns (4.39)-(4.40) *)
```

In[∘]:=
```
1/Sqrt[Δp];
Series[%, {t, 0, 4}]
Sqrt[Δ0 Δm];
Series[%, {t, 0, 4}]
```

Out[∘]= $1 + x\,t + x^2\,t^2 + x^3\,t^3 + \frac{1}{8}\left(48\,x + 8\,x^4\right)t^4 + O[t]^{9/2}$

Out[∘]= $1 - \frac{2\,t^2}{x} - 4\,t^3 - \frac{2\,t^4}{x^2} + O[t]^5$

Section 4.6

In[∘]:=
```
(* we now wish to take eqn (4.26), divide by Sqrt[Δ+],
and take the [x^>] and [x^<] parts of that *)
(* we must divide by x first, otherwise we end up with the term Q4,4
 which cannot be reduced to a combination of Q[0,0], Q0,1 and Q1,0 *)
(* sadly this makes the calculations more complicated *)
```

In[∘]:=
```
(* it is simpler to leave the Xi unevaluated until we need them *)
(* so define *)
ΔΔ0 = t^2 XX2 XX3;
ΔΔp = (1 - x/XX2) (1 - x/XX3);
ΔΔm = 1 - XX1/x;
```

In[∘]:=
```
(* the following two expansions will be useful *)
(* the expansion of 1/Sqrt[Δ+] *)
Series[1/Sqrt[ΔΔp], {x, 0, 5}];
ApplyToSeries[Factor, %]
(* and the expansion of Sqrt[Δ_] *)
Series[Sqrt[ΔΔm], {x, Infinity, 5}]
```

Out[∘]= $1 + \frac{(XX_2 + XX_3)\,x}{2\,XX_2\,XX_3} + \frac{\left(3\,XX_2^2 + 2\,XX_2\,XX_3 + 3\,XX_3^2\right)x^2}{8\,XX_2^2\,XX_3^2} + \frac{(XX_2 + XX_3)\left(5\,XX_2^2 - 2\,XX_2\,XX_3 + 5\,XX_3^2\right)x^3}{16\,XX_2^3\,XX_3^3} + \frac{\left(35\,XX_2^4 + 20\,XX_2^3\,XX_3 + 18\,XX_2^2\,XX_3^2 + 20\,XX_2\,XX_3^3 + 35\,XX_3^4\right)x^4}{128\,XX_2^4\,XX_3^4} + \frac{(XX_2 + XX_3)\left(63\,XX_2^4 - 28\,XX_2^3\,XX_3 + 58\,XX_2^2\,XX_3^2 - 28\,XX_2\,XX_3^3 + 63\,XX_3^4\right)x^5}{256\,XX_2^5\,XX_3^5} + O[x]^6$

Out[∘]= $1 - \frac{XX_1}{2\,x} - \frac{XX_1^2}{8\,x^2} - \frac{XX_1^3}{16\,x^3} - \frac{5\,XX_1^4}{128\,x^4} - \frac{7\,XX_1^5}{256\,x^5} + O\left[\frac{1}{x}\right]^6$

In[∘]:=
```
(* first take the [x^>] part *)
(* first the Q[x,0] term *)
(* need to remove the x^0 part *)
μx,0 / x / Sqrt[ΔΔp] * Q[x, 0] -
  Coefficient[Expand[μx,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, 0] * Q[0, 0] -
  Coefficient[Expand[μx,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x * (Q[0, 0] + Q1,0 * x);
xposLHS1 = Collect[%, {Q[__], Q1,0}, Factor]
(* check it *)
μx,0 / x / Sqrt[Δp] * Q[x, 0] /. {Q[x, 0] → QQcy[9, 0]};
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposLHS1 /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  {Q[x, 0] → QQcy[9, 0], Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0]};
% - %% // Simplify
```

Out[∘]=
$$-4 a^3 (-1+b) b c Q_{1,0} t^4 + \frac{1}{x XX_2 XX_3} 2 a c t^2 \left(a^2 b t^2 x XX_2 - a^2 b^2 t^2 x XX_2 + a^2 b t^2 x XX_3 - a^2 b^2 t^2 x XX_3 + 2 a^2 b t^2 XX_2 XX_3 - 2 a^2 b^2 t^2 XX_2 XX_3 + 2 a x XX_2 XX_3 - a^2 x XX_2 XX_3 + 2 b x XX_2 XX_3 - 5 a b x XX_2 XX_3 + a^2 b x XX_2 XX_3 - 2 b^2 x XX_2 XX_3 + 3 a b^2 x XX_2 XX_3 + 2 a^2 b^2 t^3 x XX_2 XX_3\right) Q[0,0] - \frac{1}{x \sqrt{\frac{(x-XX_2)(x-XX_3)}{XX_2 XX_3}}} 2 c (1-b+b t x) (a^2 t^2 + x - a x - a t x^2 + a^2 t x^2) (2 a b t^2 + 2 x - a x - b x - a t x^2 - b t x^2 + 2 a b t x^2) Q[x,0]$$

Out[∘]=
$$O[t]^{19/2}$$

In[•]:=
```
(* then the non-Q terms *)
μ / x / Sqrt[ΔΔp] - Coefficient[Expand[μ / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, 0] -
   Coefficient[Expand[μ / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x;
Factor[CoefficientList[ν, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν, x]] - 1}]].
   Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 3}] + O[x, Infinity] * x^(3 - n)],
    {n, 1, Length[CoefficientList[ν, x]]}];
xposRHS1 = %% + % * Sqrt[ΔΔ0] / x
(* check it *)
(μ + ν Sqrt[Δ]) / x / Sqrt[Δp];
Series[%, {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposRHS1 /. {XX1 → X1, XX2 → X2, XX3 → X3};
Series[%, {t, 0, 9}];
% - %%% // Simplify
```

Out[•]=
$$-a^3 b t^2 - a^2 b^2 t^2 + 2 a^3 b^2 t^2 - \frac{2 a^3 b^2 t^4}{x} - \frac{a^3 b^2 t^4}{XX_2} + \frac{1}{x \sqrt{\left(1 - \frac{x}{XX_2}\right)\left(1 - \frac{x}{XX_3}\right)}} a \left(2 a b t^2 + 2 x - a x - b x - a t x^2 - b t x^2 + 2 a b t x^2\right) \left(a b t^2 - x + a x + b x - a b x - 2 b t x^2 + a b t x^2 + a b t^2 x^3\right) - \frac{a^3 b^2 t^4}{XX_3} + \frac{1}{x}\left(a \left(a - a^2 + b - 2 a b + a^2 b - b^2 + a b^2 + 2 a^2 b^2 t^3\right) x^2 - a^2 b \left(-a - b + 2 a b\right) t x^3 \left(1 - \frac{XX_1}{2 x}\right) + a^2 b \left(-a - b + 2 a b\right) t^2 x^4 \left(1 - \frac{XX_1}{2 x} - \frac{XX_1^2}{8 x^2}\right)\right) \sqrt{t^2 XX_2 XX_3}$$

Out[•]= $O[t]^{19/2}$

In[∘]:=
```
(* then the Q[0,0] terms *)
μ0,0 / x / Sqrt[ΔΔp] - Coefficient[Expand[μ0,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, 0] -
  Coefficient[Expand[μ0,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x;
Factor[CoefficientList[ν0,0, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν0,0, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 3}] + O[x, Infinity] * x^(3 - n)],
   {n, 1, Length[CoefficientList[ν0,0, x]]}];
xposRHS2 = (%% + % * Sqrt[ΔΔ0] / x) * Q[0, 0]
(* check it *)
(μ0,0 + ν0,0 Sqrt[Δ]) / x / Sqrt[Δp];
Series[%, {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposRHS2 / Q[0, 0] /. {XX1 → X1, XX2 → X2, XX3 → X3};
Series[%, {t, 0, 9}];
% - %%% // Simplify
```

Out[∘]=

$$\left(a^3 b t^2 + a^2 b^2 t^2 - 2 a^3 b^2 t^2 + 2 a^2 c t^2 - a^3 c t^2 + 2 a b c t^2 - 6 a^2 b c t^2 + a^3 b c t^2 - 3 a b^2 c t^2 + 5 a^2 b^2 c t^2 + 2 a^3 b^2 c t^5 - \frac{-2 a^3 b^2 t^4 - 2 a^3 b c t^4 + 2 a^2 b^2 c t^4 + 2 a^3 b^2 c t^4}{x} + \frac{a^3 b^2 t^4}{XX_2} + \frac{a^3 b c t^4}{XX_2} - \frac{a^2 b^2 c t^4}{XX_2} - \frac{a^3 b^2 c t^4}{XX_2} - \frac{1}{x \sqrt{\left(1 - \frac{x}{XX_2}\right)\left(1 - \frac{x}{XX_3}\right)}} \left(2 a b t^2 + 2 x - a x - b x - a t x^2 - b t x^2 + 2 a b t x^2\right) \left(a^2 b t^2 + a^2 c t^2 - a b c t^2 - a^2 b c t^2 - a x + a^2 x + a b x - a^2 b x + 2 c x - 2 a c x - 2 b c x + 2 a b c x + a^2 b c t^3 x - 2 a b t x^2 + a^2 b t x^2 + a^2 c t x^2 + 2 b c t x^2 - a b c t x^2 - a^2 b c t x^2 + a^2 b t^2 x^3 - a^2 c t^2 x^3 - a b c t^2 x^3 + a^2 b c t^2 x^3\right) + \frac{a^3 b^2 t^4}{XX_3} + \frac{a^3 b c t^4}{XX_3} - \frac{a^2 b^2 c t^4}{XX_3} - \frac{a^3 b^2 c t^4}{XX_3} + \frac{1}{x} \left(-a \left(a - a^2 + b - 2 a b + a^2 b - b^2 + a b^2 - 2 c + 2 a c + 2 b c - 2 a b c + 2 a^2 b^2 t^3 - a^2 b c t^3 - a b^2 c t^3\right) x^2 + a \left(-a^2 b - a b^2 + 2 a^2 b^2 + a^2 c - a^2 b c + b^2 c - a b^2 c\right) t x^3 \left(1 - \frac{XX_1}{2 x}\right) - a \left(-a^2 b - a b^2 + 2 a^2 b^2 + a^2 c - a^2 b c + b^2 c - a b^2 c\right) t^2 x^4 \left(1 - \frac{XX_1}{2 x} - \frac{XX_1^2}{8 x^2}\right)\right) \sqrt{t^2 XX_2 XX_3}\right) Q[0, 0]$$

Out[∘]= $O[t]^{19/2}$

In[•]:=
```
(* then the Q0,1 terms *)
μ0,1 / x / Sqrt[ΔΔp] - Coefficient[Expand[μ0,1 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, 0] -
  Coefficient[Expand[μ0,1 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x;
Factor[CoefficientList[ν0,1, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν0,1, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 3}] + O[x, Infinity] * x^(3 - n)],
   {n, 1, Length[CoefficientList[ν0,1, x]]}];
xposRHS3 = (%% + % * Sqrt[ΔΔ0] / x) * Q0,1
(* check it *)
(μ0,1 + ν0,1 Sqrt[Δ]) / x / Sqrt[Δp];
Series[%, {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposRHS3 / Q0,1 /. {XX1 → X1, XX2 → X2, XX3 → X3};
Series[%, {t, 0, 9}];
% - %%% // Simplify
```

Out[•]=
$$Q_{0,1}\left(-2\,a^2\,b^2\,c\,t^4+2\,a^3\,b^2\,c\,t^4-\frac{(-1+a)\,a\,b\,c\,t^2\left(2\,a\,b\,t^2+2\,x-a\,x-b\,x-a\,t\,x^2-b\,t\,x^2+2\,a\,b\,t\,x^2\right)}{\sqrt{\left(1-\frac{x}{XX_2}\right)\left(1-\frac{x}{XX_3}\right)}}+(-1+a)\,a\,(a-b)\,b\,c\,t^2\,x\,\sqrt{t^2\,XX_2\,XX_3}\right)$$

Out[•]=
$$O[t]^{10}$$

In[∘]:=
```
(* then the Q1,0 terms *)
μ1,0 / x / Sqrt[ΔΔp] - Coefficient[Expand[μ1,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, 0] -
  Coefficient[Expand[μ1,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x;
Factor[CoefficientList[ν1,0, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν1,0, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 3}] + O[x, Infinity] * x^(3 - n)],
   {n, 1, Length[CoefficientList[ν1,0, x]]}];
xposRHS4 = (%% + % * Sqrt[ΔΔ0] / x) * Q1,0
(* check it *)
(μ1,0 + ν1,0 Sqrt[Δ]) / x / Sqrt[Δp];
Series[%, {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposRHS4 / Q1,0 /. {XX1 → X1, XX2 → X2, XX3 → X3};
Series[%, {t, 0, 9}];
% - %%% // Simplify
```

Out[∘]=
$$Q_{1,0}\left(2\,a^3\,b\,c\,t^4-2\,a^3\,b^2\,c\,t^4+\frac{a^2\,(-1+b)\,c\,t^2\,\left(2\,a\,b\,t^2+2\,x-a\,x-b\,x-a\,t\,x^2-b\,t\,x^2+2\,a\,b\,t\,x^2\right)}{\sqrt{\left(1-\frac{x}{XX_2}\right)\left(1-\frac{x}{XX_3}\right)}}-a^2\,(a-b)\,(-1+b)\,c\,t^2\,x\,\sqrt{t^2\,XX_2\,XX_3}\right)$$

Out[∘]= $O[t]^{10}$

In[•]:=
```
(* and finally (the most complicated) the Q0d[1/x] term *)
Factor[
   CoefficientList[ν0d, x] * Table[x^n, {n, 0, Length[CoefficientList[ν0d, x]] - 1}].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
   {n, 1, Length[CoefficientList[ν0d, x]]}]];
CoefficientList[%, x] * Table[x^n, {n, 0, Length[CoefficientList[%, x]] - 1}]
(* because I've symbolised Qi,j this last thing has to be done manually *)
Length[%]
(%%.{0, 0, Q[0, 0], Q[0, 0] + Q1,1/x, Q[0, 0] + Q1,1/x + Q2,2/x^2}) * Sqrt[ΔΔ0];
(* now do some eliminations *)
xposLHS2 = % / x /. Solve[Q11eqn == 0, Q2,2][[1]] /. Solve[Q00eqn == 0, Q1,1][[1]] /.
   Solve[Q10eqn == 0, Q2,1][[1]] /. Solve[Q01eqn == 0, Q1,2][[1]]
(* check it *)
ν0d Sqrt[Δ] / Sqrt[Δp] / x * Q0d[1/x] /. {Q0d[1/x] → QQdkeval[9, 0, 1 / x]};
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposLHS2 /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]};
% - %% // Simplify
```

Out[•]=
$$\left\{0,\ x\left(-2a\left(-a^2+a^2b-b^2+ab^2\right)ct^2-ac\left(1-a-b+ab-a^2bt^3-ab^2t^3+2a^2b^2t^3\right)XX_1+\frac{1}{4}(-1+a)a(-1+b)(a+b)ct\,XX_1^2+\frac{1}{8}(1-a)a^2(-1+b)bct^2XX_1^3\right),\right.$$
$$x^2\left(2ac\left(1-a-b+ab-a^2bt^3-ab^2t^3+2a^2b^2t^3\right)+(-1+a)a(-1+b)(a+b)ct\,XX_1+\frac{1}{4}(1-a)a^2(-1+b)bct^2XX_1^2\right),$$
$$x^3\left(-2(-1+a)a(-1+b)(a+b)ct+(1-a)a^2(-1+b)bct^2XX_1\right),$$
$$\left.2(-1+a)a^2(-1+b)bct^2x^4\right\}$$

Out[•]= 5

Out[•]=
$$\frac{1}{x}\sqrt{t^2XX_2XX_3}\left(x^2\left(2ac\left(1-a-b+ab-a^2bt^3-ab^2t^3+2a^2b^2t^3\right)+(-1+a)a(-1+b)(a+b)ct\,XX_1+\frac{1}{4}(1-a)a^2(-1+b)bct^2XX_1^2\right)Q[0,0]+\right.$$
$$x^3\left(-2(-1+a)a(-1+b)(a+b)ct+(1-a)a^2(-1+b)bct^2XX_1\right)\left(\frac{-1+Q[0,0]}{c\,t\,x}+Q[0,0]\right)+$$
$$\left.2(-1+a)a^2(-1+b)bct^2x^4\left(\frac{-Q_{0,1}t-Q_{1,0}t+\frac{-1+Q[0,0]}{c\,t}}{t\,x^2}+\frac{-1+Q[0,0]}{c\,t\,x}+Q[0,0]\right)\right)$$

Out[•]= $O[t]^{19/2}$

In[•]:=
```
(* check it *)
-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3 + xposRHS4;
% /. {XX1 → X1, XX2 → X2, XX3 → X3};
% /. {Q[x, 0] → QQcy[12, 0], Q[0, 0] → QQcxy[12, 0, 0],
    Q0,1 → QQcxy[12, 0, 1], Q1,0 → QQcxy[12, 1, 0]} // Simplify
```

Out[•]= $O[t]^{25/2}$

In[•]:=
```
(* next, the [x^<] part *)
```

In[•]:=
```
(* contribution from the Q[x,0] part is easy *)
xnegLHS1 = Coefficient[Expand[μx,0 / x * (1 + (XX2 + XX3) x / (2 XX2 XX3))], x, -1] / x * Q[0, 0]
```

Out[•]= $$\frac{\left(-4 a^3 b c t^4+4 a^3 b^2 c t^4\right) Q[0,0]}{x}$$

In[•]:=
```
(* the non-Q term *)
Coefficient[Expand[μ / x], x, -1] / x;
Factor[
    CoefficientList[ν, x] * Table[x^n, {n, 0, Length[CoefficientList[ν, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
    {n, 1, Length[CoefficientList[ν, x]]}];
xnegRHS1 = %% + (ν Sqrt[ΔΔm] / x - % / x) * Sqrt[ΔΔ0]
(* check it *)
Series[(μ + ν Sqrt[Δ]) / x / Sqrt[Δp], {t, 0, 12}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS1 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 12}];
% - %% // Simplify
```

Out[•]=
$$\frac{2 a^3 b^2 t^4}{x}+\left(a\left(a^2 b t^2+a b^2 t^2-2 a^2 b^2 t^2+a x-a^2 x+b x-2 a b x+a^2 b x-b^2 x+a b^2 x+2 a^2 b^2 t^3 x+a^2 b t x^2+a b^2 t x^2-2 a^2 b^2 t x^2-a^2 b t^2 x^3-a b^2 t^2 x^3+2 a^2 b^2 t^2 x^3\right)\sqrt{1-\frac{XX_1}{x}}-\frac{1}{x}\left(-a^2 b(-a-b+2 a b) t^2 x+a\left(a-a^2+b-2 a b+a^2 b-b^2+a b^2+2 a^2 b^2 t^3\right) x^2\left(1-\frac{XX_1}{2 x}\right)-a^2 b(-a-b+2 a b) t x^3\left(1-\frac{XX_1}{2 x}-\frac{XX_1^2}{8 x^2}\right)+a^2 b(-a-b+2 a b) t^2 x^4\left(1-\frac{XX_1}{2 x}-\frac{XX_1^2}{8 x^2}-\frac{XX_1^3}{16 x^3}\right)\right)\right)\sqrt{t^2 XX_2 XX_3}$$

Out[•]= $O[t]^{25/2}$

In[∘]:=
```
(* the Q[0,0] term *)
Coefficient[Expand[μ_{0,0} / x], x, -1] / x;
Factor[CoefficientList[ν_{0,0}, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν_{0,0}, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔ_m], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
   {n, 1, Length[CoefficientList[ν_{0,0}, x]]}];
xnegRHS2 = (%% + (ν_{0,0} Sqrt[ΔΔ_m] / x - % / x) * Sqrt[ΔΔ_0]) * Q[0, 0]
(* check it *)
Series[(μ_{0,0} + ν_{0,0} Sqrt[Δ]) / x / Sqrt[Δ_p], {t, 0, 12}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS2 / Q[0, 0] /. {XX_1 → X_1, XX_2 → X_2, XX_3 → X_3}, {t, 0, 12}];
% - %% // Simplify
```

Out[∘]=

$$\Bigg(\frac{-2 a^3 b^2 t^4 - 2 a^3 b c t^4 + 2 a^2 b^2 c t^4 + 2 a^3 b^2 c t^4}{x} + \Bigg(-a\,\big(a^2 b t^2 + a b^2 t^2 - 2 a^2 b^2 t^2 - a^2 c t^2 + a^2 b c t^2 - b^2 c t^2 + a b^2 c t^2 + a x - a^2 x + b x - 2 a b x + a^2 b x - b^2 x + a b^2 x - 2 c x + 2 a c x + 2 b c x - 2 a b c x + 2 a^2 b^2 t^3 x - a^2 b c t^3 x - a b^2 c t^3 x + a^2 b t x^2 + a b^2 t x^2 - 2 a^2 b^2 t x^2 - a^2 c t x^2 + a^2 b c t x^2 - b^2 c t x^2 + a b^2 c t x^2 - a^2 b t^2 x^3 - a b^2 t^2 x^3 + 2 a^2 b^2 t^2 x^3 + a^2 c t^2 x^3 - a^2 b c t^2 x^3 + b^2 c t^2 x^3 - a b^2 c t^2 x^3\big)\sqrt{1 - \frac{XX_1}{x}} - \frac{1}{x}\Big(a\,\big(-a^2 b - a b^2 + 2 a^2 b^2 + a^2 c - a^2 b c + b^2 c - a b^2 c\big)\, t^2 x - a\,\big(a - a^2 + b - 2 a b + a^2 b - b^2 + a b^2 - 2 c + 2 a c + 2 b c - 2 a b c + 2 a^2 b^2 t^3 - a^2 b c t^3 - a b^2 c t^3\big)\, x^2 \left(1 - \frac{XX_1}{2 x}\right) + a\,\big(-a^2 b - a b^2 + 2 a^2 b^2 + a^2 c - a^2 b c + b^2 c - a b^2 c\big)\, t\, x^3 \left(1 - \frac{XX_1}{2 x} - \frac{XX_1^2}{8 x^2}\right) - a\,\big(-a^2 b - a b^2 + 2 a^2 b^2 + a^2 c - a^2 b c + b^2 c - a b^2 c\big)\, t^2 x^4 \left(1 - \frac{XX_1}{2 x} - \frac{XX_1^2}{8 x^2} - \frac{XX_1^3}{16 x^3}\right)\Big)\Bigg)\sqrt{t^2\, XX_2\, XX_3}\Bigg)\, Q[0, 0]$$

Out[∘]= $O[t]^{25/2}$

In[∘]:=
```
(* the Q0,1 term *)
Coefficient[Expand[μ0,1 / x], x, -1] / x;
Factor[CoefficientList[ν0,1, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν0,1, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
   {n, 1, Length[CoefficientList[ν0,1, x]]}];
xnegRHS3 = (%% + (ν0,1 Sqrt[ΔΔm] / x - % / x) * Sqrt[ΔΔ0]) * Q0,1
(* check it *)
Series[(μ0,1 + ν0,1 Sqrt[Δ]) / x / Sqrt[Δp], {t, 0, 12}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS3 / Q0,1 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 12}];
% - %% // Simplify
```

Out[∘]= $Q_{0,1}\left((-1+a)\,a\,(a-b)\,b\,c\,t^2\,x\sqrt{1-\frac{XX_1}{x}}-(-1+a)\,a\,(a-b)\,b\,c\,t^2\,x\left(1-\frac{XX_1}{2\,x}\right)\right)\sqrt{t^2\,XX_2\,XX_3}$

Out[∘]= $O[t]^{13}$

In[∘]:=
```
(* the Q1,0 term *)
Coefficient[Expand[μ1,0 / x], x, -1] / x;
Factor[CoefficientList[ν1,0, x] *
    Table[x^n, {n, 0, Length[CoefficientList[ν1,0, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
   {n, 1, Length[CoefficientList[ν1,0, x]]}];
xnegRHS4 = (%% + (ν1,0 Sqrt[ΔΔm] / x - % / x) * Sqrt[ΔΔ0]) * Q1,0
(* check it *)
Series[(μ1,0 + ν1,0 Sqrt[Δ]) / x / Sqrt[Δp], {t, 0, 12}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS4 / Q1,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 12}];
% - %% // Simplify
```

Out[∘]= $Q_{1,0}\left(-a^2\,(a-b)\,(-1+b)\,c\,t^2\,x\sqrt{1-\frac{XX_1}{x}}+a^2\,(a-b)\,(-1+b)\,c\,t^2\,x\left(1-\frac{XX_1}{2\,x}\right)\right)\sqrt{t^2\,XX_2\,XX_3}$

Out[∘]= $O[t]^{13}$

In[∘]:=
```
(* the Q0d[1/x] term *)
Factor[
   CoefficientList[ν0d, x] * Table[x^n, {n, 0, Length[CoefficientList[ν0d, x]] - 1}]].
  Table[Normal[Series[Sqrt[ΔΔm], {x, Infinity, n - 2}] + O[x, Infinity] * x^(2 - n)],
   {n, 1, Length[CoefficientList[ν0d, x]]}]];
CoefficientList[% / x, x] * Table[x^n,
   {n, 0, Length[CoefficientList[% / x, x]] - 1}];
(* because I've symbolised Qi,j this last thing has to be done manually *)
Length[%]
(%%.{Q[0, 0], Q[0, 0] + Q1,1 / x, Q[0, 0] + Q1,1 / x + Q2,2 / x^2,
      Q[0, 0] + Q1,1 / x + Q2,2 / x^2 + Q3,3 / x^3}) * Sqrt[ΔΔ0];
(* now do some eliminations *)
% /. Solve[Q22eqn == 0, Q3,3][[1]] /. Solve[Q11eqn == 0, Q2,2][[1]] /.
     Solve[Q00eqn == 0, Q1,1][[1]] /.
   Solve[Q10eqn == 0, Q2,1][[1]] /. Solve[Q01eqn == 0, Q1,2][[1]];
xnegLHS2 = ν0d Sqrt[ΔΔm] Sqrt[ΔΔ0] / x * Q0d[1/x] - %
(* check it *)
ν0d Sqrt[Δ] / x / Sqrt[Δp] * Q0d[1/x] /. {Q0d[1/x] → QQdkeval[9, 0, 1 / x]};
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
xnegLHS2 /. {XX1 → X1, XX2 → X2, XX3 → X3} /. {Q0d[1/x] → QQdkeval[9, 0, 1 / x],
   Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]};
% - %% // Simplify
```

Out[∘]= 4

Out[•]= $-\sqrt{t^2\,XX_2\,XX_3}\Bigg(\Big(-2\,a\,(-a^2+a^2\,b-b^2+a\,b^2)\,c\,t^2-a\,c\,(1-a-b+a\,b-a^2\,b\,t^3-a\,b^2\,t^3+2\,a^2\,b^2\,t^3)\,XX_1+\frac{1}{4}\,(-1+a)\,a\,(-1+b)\,(a+b)\,c\,t\,XX_1^2-\frac{1}{8}\,(-1+a)\,a^2\,(-1+b)\,b\,c\,t^2\,XX_1^3\Big)\,Q[0,0]+x\,\Big(2\,a\,c\,(1-a-b+a\,b-a^2\,b\,t^3-a\,b^2\,t^3+2\,a^2\,b^2\,t^3)+(-1+a)\,a\,(-1+b)\,(a+b)\,c\,t\,XX_1+\frac{1}{4}\,(1-a)\,a^2\,(-1+b)\,b\,c\,t^2\,XX_1^2\Big)\,\Big(\frac{-1+Q[0,0]}{c\,t\,x}+Q[0,0]\Big)+x^2\,\big(-2\,(-1+a)\,a\,(-1+b)\,(a+b)\,c\,t+(1-a)\,a^2\,(-1+b)\,b\,c\,t^2\,XX_1\big)\Bigg(\frac{-Q_{0,1}\,t-Q_{1,0}\,t+\frac{-1+Q[0,0]}{c\,t}}{t\,x^2}+\frac{-1+Q[0,0]}{c\,t\,x}+Q[0,0]\Bigg)+2\,(-1+a)\,a^2\,(-1+b)\,b\,c\,t^2\,x^3\Bigg(\frac{-Q_{0,1}\,t-Q_{1,0}\,t+\frac{-1+Q[0,0]}{c\,t}}{t\,x^2}+\frac{-1+Q[0,0]}{c\,t\,x}+Q[0,0]+\frac{\frac{-Q_{0,1}\,t-Q_{1,0}\,t+\frac{-1+Q[0,0]}{c\,t}}{t}-\frac{Q_{1,0}-a\,t\,Q[0,0]}{a}-\frac{Q_{0,1}-b\,t\,Q[0,0]}{b}}{t\,x^3}\Bigg)\Bigg)+\frac{1}{x}2\,a\,c\,(a^2\,t^2+x-a\,x-a\,t\,x^2+a^2\,t\,x^2)\,(b^2\,t^2+x-b\,x-b\,t\,x^2+b^2\,t\,x^2)\,\sqrt{1-\frac{XX_1}{x}}\,\sqrt{t^2\,XX_2\,XX_3}\,Q_0^d\Big[\frac{1}{x}\Big]$

Out[•]= $O[t]^9$

```
In[•]:= (* check it *)
-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3 + xnegRHS4;
% /. {XX1 → X1, XX2 → X2, XX3 → X3};
% /. {Q0d[1/x] → QQdkeval[12, 0, 1/x], Q[0, 0] → QQcxy[12, 0, 0],
    Q0,1 → QQcxy[12, 0, 1], Q1,0 → QQcxy[12, 1, 0]} // Simplify
```

Out[•]= $O[t]^{12}$

```
In[•]:= (* constructing eqn (4.41) *)
```

In[•]:=
```
Px,0 = (a^2 t^2 + x - a x - a t x^2 + a^2 t x^2) (2 a b t^2 + 2 x - a x - b x - a t x^2 - b t x^2 + 2 a b t x^2)
σx,0 = Coefficient[xposLHS1, Q[x, 0]] / Px,0
(* and then, without bothering to try simplifying anything, *)
σ0,0 = Coefficient[
  - xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3 + xposRHS4, Q[0, 0]];
σ0,1 = Coefficient[- xposLHS1 - xposLHS2 + xposRHS1 +
   xposRHS2 + xposRHS3 + xposRHS4, Q0,1];
σ1,0 = Coefficient[- xposLHS1 - xposLHS2 + xposRHS1 +
   xposRHS2 + xposRHS3 + xposRHS4, Q1,0];
σ = (- xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3 + xposRHS4 /.
   {Q[x, 0] → 0, Q[0, 0] → 0, Q0,1 → 0, Q1,0 → 0});
(* check it *)
- σx,0 Px,0 * Q[x, 0] + σ + σ0,0 * Q[0, 0] + σ0,1 * Q0,1 + σ1,0 * Q1,0 /.
  {XX1 → X1, XX2 → X2, XX3 → X3} /. {Q[x, 0] → QQcy[9, 0], Q[0, 0] → QQcxy[9, 0, 0],
  Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} // Simplify
```

Out[•]= $\left(a^2 t^2 + x - a x - a t x^2 + a^2 t x^2\right) \left(2 a b t^2 + 2 x - a x - b x - a t x^2 - b t x^2 + 2 a b t x^2\right)$

Out[•]= $-\dfrac{2 c \left(1 - b + b t x\right)}{x \sqrt{\frac{(x - XX_2)(x - XX_3)}{XX_2 XX_3}}}$

Out[•]= $O[t]^{19/2}$

In[•]:=
```
(* constructing eqn (4.42) *)
P0^d = (- a t + a^2 t + x - a x + a^2 t^2 x^2) (- b t + b^2 t + x - b x + b^2 t^2 x^2)
τ0^d = (Coefficient[xnegLHS2 / x^3 /. x → 1 / x, Q0^d[x]] // Factor) / P0^d
(* and then *)
τ0,0 = Coefficient[
  (- xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3 + xnegRHS4) / x^3 /. x → 1 / x,
  Q[0, 0]];
τ0,1 = Coefficient[(- xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3 + xnegRHS4) /
    x^3 /. x → 1 / x, Q0,1];
τ1,0 = Coefficient[(- xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3 + xnegRHS4) /
    x^3 /. x → 1 / x, Q1,0];
τ = ((- xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3 + xnegRHS4) / x^3 /.
    x → 1 / x /. {Q0^d[x] → 0, Q[0, 0] → 0, Q0,1 → 0, Q1,0 → 0});
(* check it *)
- τ0^d P0^d * Q0^d[x] + τ + τ0,0 * Q[0, 0] + τ0,1 * Q0,1 + τ1,0 * Q1,0 /.
  {XX1 → X1, XX2 → X2, XX3 → X3} /. {Q0^d[x] → QQdk[9, 0], Q[0, 0] → QQcxy[9, 0, 0],
  Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} // Simplify
```

Out[•]= $\left(- a t + a^2 t + x - a x + a^2 t^2 x^2\right) \left(- b t + b^2 t + x - b x + b^2 t^2 x^2\right)$

Out[•]= $2 a c \sqrt{1 - x XX_1} \sqrt{t^2 XX_2 XX_3}$

Out[•]= $O[t]^{9}$

In[•]:=
```
(* the kernel roots in (4.45)-(4.46) *)
x1 = (-Sqrt[-4 a^4 t^3 + 4 a^3 t^3 + a^2 - 2 a + 1] + a - 1) / (2 a t (a - 1));
x2 = (Sqrt[-4 a^4 t^3 + 4 a^3 t^3 + a^2 - 2 a + 1] + a - 1) / (2 a t (a - 1));
x3 = (-Sqrt[(a + b - 2) ^2 - 8 a b t^3 (2 a b - a - b)] + a + b - 2) / (2 t (2 a b - a - b));
x4 = (Sqrt[(a + b - 2) ^2 - 8 a b t^3 (2 a b - a - b)] + a + b - 2) / (2 t (2 a b - a - b));
(* so that *)
{P_{x,0} /. x → x1, P_{x,0} /. x → x2, P_{x,0} /. x → x3, P_{x,0} /. x → x4} // Simplify
```

Out[•]= $\{0, 0, 0, 0\}$

In[•]:=
```
(* the kernel roots in (4.47)-(4.48) *)
x5 = (-Sqrt[-4 a^4 t^3 + 4 a^3 t^3 + a^2 - 2 a + 1] + a - 1) / (2 a^2 t^2);
x6 = (Sqrt[-4 a^4 t^3 + 4 a^3 t^3 + a^2 - 2 a + 1] + a - 1) / (2 a^2 t^2);
x7 = (-Sqrt[-4 b^4 t^3 + 4 b^3 t^3 + b^2 - 2 b + 1] + b - 1) / (2 b^2 t^2);
x8 = (Sqrt[-4 b^4 t^3 + 4 b^3 t^3 + b^2 - 2 b + 1] + b - 1) / (2 b^2 t^2);
(* so that *)
{P^d_0 /. x → x5, P^d_0 /. x → x6, P^d_0 /. x → x7, P^d_0 /. x → x8} // Simplify
```

Out[•]= $\{0, 0, 0, 0\}$

In[•]:=
```
(* and verifying which terms are power series *)
(* which one of these is a power series depends on the sign of (a-1) *)
Series[{x1, x2}, {t, 0, 2}]
(* which one of these is a power series depends on the sign of (a+b-2) *)
Series[{x3, x4}, {t, 0, 2}]
(* which one of these is a power series depends on the sign of (a-1) *)
Series[{x5, x6}, {t, 0, 1}]
(* which one of these is a power series depends on the sign of (b-1) *)
Series[{x7, x8}, {t, 0, 1}]
```

Out[•]= $$\left\{\frac{-1-\sqrt{(-1+a)^2}+a}{2\,(-1+a)\,a\,t}+\frac{a^2\,t^2}{\sqrt{(-1+a)^2}}+O[t]^3,\ \frac{-1+\sqrt{(-1+a)^2}+a}{2\,(-1+a)\,a\,t}-\frac{a^2\,t^2}{\sqrt{(-1+a)^2}}+O[t]^3\right\}$$

Out[•]= $$\left\{\frac{-2+a+b-\sqrt{(-2+a+b)^2}}{2\,(-a-b+2\,a\,b)\,t}+\frac{2\,a\,b\,t^2}{\sqrt{(-2+a+b)^2}}+O[t]^3,\right.$$
$$\left.\frac{-2+a+b+\sqrt{(-2+a+b)^2}}{2\,(-a-b+2\,a\,b)\,t}-\frac{2\,(a\,b)\,t^2}{\sqrt{(-2+a+b)^2}}+O[t]^3\right\}$$

Out[•]= $$\left\{\frac{-1-\sqrt{(-1+a)^2}+a}{2\,a^2\,t^2}+\frac{(-1+a)\,a\,t}{\sqrt{(-1+a)^2}}+O[t]^2,\ \frac{-1+\sqrt{(-1+a)^2}+a}{2\,a^2\,t^2}-\frac{(-1+a)\,a\,t}{\sqrt{(-1+a)^2}}+O[t]^2\right\}$$

Out[•]= $$\left\{\frac{-1-\sqrt{(-1+b)^2}+b}{2\,b^2\,t^2}+\frac{(-1+b)\,b\,t}{\sqrt{(-1+b)^2}}+O[t]^2,\ \frac{-1+\sqrt{(-1+b)^2}+b}{2\,b^2\,t^2}-\frac{(-1+b)\,b\,t}{\sqrt{(-1+b)^2}}+O[t]^2\right\}$$

```
In[•]:= (* these will be useful *)
Clear[xs1, xs2, xs3, xs4, xs5, xs6, xs7, xs8, Xs1, Xs2, Xs3]
xs1[n_] := xs1[n] =
  ApplyToSeries[Factor[Simplify[#, Assumptions → a > 1]] &, Series[x1, {t, 0, n}]]
xs2[n_] := xs2[n] = ApplyToSeries[
   Factor[Simplify[#, Assumptions → 0 < a < 1]] &, Series[x2, {t, 0, n}]]
xs3[n_] := xs3[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → a + b > 2]] &,
   Series[x3, {t, 0, n}]]
xs4[n_] := xs4[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < a + b < 2]] &,
   Series[x4, {t, 0, n}]]
xs5[n_] := xs5[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → a > 1]] &,
   Series[x5, {t, 0, n}]]
xs6[n_] := xs6[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < a < 1]] &,
   Series[x6, {t, 0, n}]]
xs7[n_] := xs7[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → b > 1]] &,
   Series[x7, {t, 0, n}]]
xs8[n_] := xs8[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < b < 1]] &,
   Series[x8, {t, 0, n}]]
Xs1[n_] := Xs1[n] = Series[X1, {t, 0, n}]
Xs2[n_] := Xs2[n] = Series[X2, {t, 0, n}]
Xs3[n_] := Xs3[n] = Series[X3, {t, 0, n}]
```

In[•]:=
```
(* now we can evaluate the coefficients in
  (4.41) and (4.42) after cancelling the kernels *)
(* this gives the ξ^(i) coefficients in (4.49) *)
Clear[Hx1, Hx3, Hx5, Hx7]
Hx1[n_] := Hx1[n] = Simplificate /@ (ApplyToSeries[Simplify, #] & /@
    ({σ, σ0,0, σ0,1, σ1,0} /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} /. x → xs1[n]))
Hx3[n_] := Hx3[n] = Simplificate /@ (ApplyToSeries[Simplify, #] & /@
    ({σ, σ0,0, σ0,1, σ1,0} /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} /. x → xs3[n]))
Hx5[n_] := Hx5[n] = Simplificate /@ (ApplyToSeries[Simplify, #] & /@
    ({τ, τ0,0, τ0,1, τ1,0} /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} /. x → xs5[n]))
Hx7[n_] := Hx7[n] = Simplificate /@ (ApplyToSeries[Simplify, #] & /@
    ({τ, τ0,0, τ0,1, τ1,0} /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} /. x → xs7[n]))
(* the leading order terms *)
ApplyToSeries[Factor, #] & /@ Hx1[3]
ApplyToSeries[Factor, #] & /@ Hx3[3]
ApplyToSeries[Factor, #] & /@ Hx5[3]
ApplyToSeries[Factor, #] & /@ Hx7[3]
```

Out[•]= $\{2 a^3 (-1+b)(-1+a b) t^2+O[t]^4, -2 (a^3 (-1+b)(-1+a b)) t^2+O[t]^4, 2 a^3 b (-1+a b) c\, t^4+O[t]^6, 2 a^3 (-1+b)(-a+2 b+a b) c\, t^4+O[t]^6\}$

Out[•]= $\left\{\frac{4 (-1+a) a^2 (-1+b) b (-1+a b) t^2}{-2+a+b}+O[t]^4, -\frac{4 ((-1+a) a^2 (-1+b) b (-1+a b)) t^2}{-2+a+b}+O[t]^4, \frac{4 (-1+a) a^2 b^2 (-1+a b) c\, t^4}{-2+a+b}+O[t]^6, \frac{4 a^3 (-1+b) b (-1+a b) c\, t^4}{-2+a+b}+O[t]^6\right\}$

Out[•]= $\{-2 ((-1+a) a^5 (-1+b)^2) t^2+O[t]^{7/2}, 2 (-1+a) a^5 (-1+b)^2 t^2+O[t]^{7/2}, -2 ((-1+a) a^5 (-1+b) b c) t^4+O[t]^{11/2}, -2 ((-1+a) a^4 (-1+b)(-a+b+a b) c) t^4+O[t]^{11/2}\}$

Out[•]= $\{-2 ((-1+a)^2 a (-1+b) b^4) t^2+O[t]^{7/2}, 2 (-1+a)^2 a (-1+b) b^4 t^2+O[t]^{7/2}, -2 ((-1+a) a (-1+b) b^3 (a-b+a b) c) t^4+O[t]^{11/2}, -2 ((-1+a) a^2 (-1+b) b^4 c) t^4+O[t]^{11/2}\}$

In[•]:=
```
(* and indeed we can verify that cancelling the kernel works *)
Hx1[9].{1, Q[0, 0], Q0,1, Q1,0};
% /. {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} //
  Simplify;
% // Simplificate
Hx3[9].{1, Q[0, 0], Q0,1, Q1,0};
% /. {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} //
  Simplify;
% // Simplificate
Hx5[9].{1, Q[0, 0], Q0,1, Q1,0};
% /. {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} //
  Simplify;
% // Simplificate
Hx7[9].{1, Q[0, 0], Q0,1, Q1,0};
% /. {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} //
  Simplify;
% // Simplificate
```

Out[•]= $O[t]^{10}$

Out[•]= $O[t]^{10}$

Out[•]= $O[t]^{19/2}$

Out[•]= $O[t]^{19/2}$

In[•]:=
```
(* then looking at the coefficient matrices *)
(* which combinations give independent equations? *)
(* these are the determinants in eqns (4.50)-(4.53) *)
Drop[#, 1] & /@ {Hx1[3], Hx3[3], Hx5[3]};
ApplyToSeries[Factor, Det[%]]
Drop[#, 1] & /@ {Hx1[3], Hx3[3], Hx7[3]};
ApplyToSeries[Factor, Det[%]]
Drop[#, 1] & /@ {Hx1[3], Hx5[3], Hx7[3]};
ApplyToSeries[Factor, Det[%]]
Drop[#, 1] & /@ {Hx3[3], Hx5[3], Hx7[3]};
ApplyToSeries[Factor, Det[%]]
```

Out[•]= $O[t]^{23/2}$

Out[•]= $\frac{16\,(-2+a)\,(-1+a)\,a^6\,(a-b)^2\,(-1+b)^2\,b^4\,(-1+a\,b)\,c^2\,t^{10}}{-2+a+b}+O[t]^{23/2}$

Out[•]= $-8\,\left((-1+a)^2\,a^8\,(a-b)^2\,(-1+b)^2\,b^3\,(1-a+a\,b)\,c^2\right)\,t^{10}+O[t]^{23/2}$

Out[•]= $-\frac{16\,\left((-1+a)^2\,a^7\,(a-b)^2\,(-1+b)^2\,b^4\,(-1+a\,b)\,c^2\right)\,t^{10}}{-2+a+b}+O[t]^{23/2}$

```
In[•]:= (* try the first one again with higher powers *)
        Drop[#, 1] & /@ {Hx1[9], Hx3[9], Hx5[9]};
        ApplyToSeries[Factor, Det[%]]
```

Out[•]= $O[t]^{35/2}$

```
In[•]:= (* finally, verify that we do indeed get a solution out at the end *)
        Inverse[Drop[#, 1] & /@ {Hx1[9], Hx3[9], Hx7[9]}].
            (-Drop[#, -3] & /@ {Hx1[9], Hx3[9], Hx7[9]}) // Simplify // Flatten;
        ApplyToSeries[Expand, #] & /@ %
        ({Q[0, 0], Q0,1, Q1,0} /.
            {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]}) - %
        Inverse[Drop[#, 1] & /@ {Hx1[9], Hx5[9], Hx7[9]}].
            (-Drop[#, -3] & /@ {Hx1[9], Hx5[9], Hx7[9]}) // Simplify // Flatten;
        ApplyToSeries[Expand, #] & /@ %
        ({Q[0, 0], Q0,1, Q1,0} /.
            {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]}) - %
        Inverse[Drop[#, 1] & /@ {Hx3[9], Hx5[9], Hx7[9]}].
            (-Drop[#, -3] & /@ {Hx3[9], Hx5[9], Hx7[9]}) // Simplify // Flatten;
        ApplyToSeries[Expand, #] & /@ %
        ({Q[0, 0], Q0,1, Q1,0} /.
            {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]}) - %
```

Out[•]= $\{1 + (a\,c + b\,c)\,t^3 + (2\,a\,c + 2\,a^2\,c + a^3\,c + 2\,b\,c + 2\,a\,b\,c + 2\,b^2\,c + b^3\,c + a^2\,c^2 + 2\,a\,b\,c^2 + b^2\,c^2)\,t^6 + O[t]^{15/2},\ b\,t + (a\,b + b^2 + b^3 + a\,b\,c + b^2\,c)\,t^4 + O[t]^{11/2},\ a\,t + (a^2 + a^3 + a\,b + a^2\,c + a\,b\,c)\,t^4 + O[t]^{11/2}\}$

Out[•]= $\{O[t]^{15/2}, O[t]^{11/2}, O[t]^{11/2}\}$

Out[•]= $\{1 + (a\,c + b\,c)\,t^3 + (2\,a\,c + 2\,a^2\,c + a^3\,c + 2\,b\,c + 2\,a\,b\,c + 2\,b^2\,c + b^3\,c + a^2\,c^2 + 2\,a\,b\,c^2 + b^2\,c^2)\,t^6 + O[t]^{15/2},\ b\,t + (a\,b + b^2 + b^3 + a\,b\,c + b^2\,c)\,t^4 + O[t]^{11/2},\ a\,t + (a^2 + a^3 + a\,b + a^2\,c + a\,b\,c)\,t^4 + O[t]^{11/2}\}$

Out[•]= $\{O[t]^{15/2}, O[t]^{11/2}, O[t]^{11/2}\}$

Out[•]= $\{1 + (a\,c + b\,c)\,t^3 + (2\,a\,c + 2\,a^2\,c + a^3\,c + 2\,b\,c + 2\,a\,b\,c + 2\,b^2\,c + b^3\,c + a^2\,c^2 + 2\,a\,b\,c^2 + b^2\,c^2)\,t^6 + O[t]^{15/2},\ b\,t + (a\,b + b^2 + b^3 + a\,b\,c + b^2\,c)\,t^4 + O[t]^{11/2},\ a\,t + (a^2 + a^3 + a\,b + a^2\,c + a\,b\,c)\,t^4 + O[t]^{11/2}\}$

Out[•]= $\{O[t]^{15/2}, O[t]^{11/2}, O[t]^{11/2}\}$

Section 4.7

```
In[•]:= (* constructing another equation using the [x^0] part of eqn (4.26) *)
(* since we already have the positive and negative parts,
we can just subtract them away *)
(* the LHS *)
μx,0 / x / Sqrt[ΔΔp] * Q[x, 0] + ν0d Sqrt[ΔΔm] Sqrt[ΔΔ0] / x * Q0d[1/x] -
  xposLHS1 - xposLHS2 - xnegLHS1 - xnegLHS2;
x0LHS = Collect[%, {Q[0, 0], Q1,0, Q0,1}, Factor]
(* check it *)
μx,0 / x / Sqrt[Δp] * Q[x, 0] + ν0d Sqrt[Δm] Sqrt[Δ0] / x * Q0d[1/x] /.
  {Q[x, 0] → QQcy[9, 0], Q0d[1/x] → QQdkeval[9, 0, 1 / x]};
ApplyToSeries[Coefficient[#, x, 0] &, %];
x0LHS /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0], Q0,1 → QQcxy[9, 0, 1]};
% - %% // Simplify
```

Out[ ]=

$$(-1+a)\,a\,(-1+b)\,b\,c\,Q_{0,1}\,t\,(2-2\,a+a\,t\,XX_1)\,\sqrt{t^2\,XX_2\,XX_3}\,-$$

$$\frac{1}{4\,t}\,a\,\big(8-16\,a+8\,a^2-16\,b+32\,a\,b-16\,a^2\,b+8\,b^2-16\,a\,b^2+8\,a^2\,b^2-8\,a^2\,b\,t^3-8\,a\,b^2\,t^3+$$

$$16\,a^2\,b^2\,t^3+4\,a\,t\,XX_1-4\,a^2\,t\,XX_1+4\,b\,t\,XX_1-12\,a\,b\,t\,XX_1+8\,a^2\,b\,t\,XX_1-4\,b^2\,t\,XX_1+$$

$$8\,a\,b^2\,t\,XX_1-4\,a^2\,b^2\,t\,XX_1-a\,b\,t^2\,XX_1^2+a^2\,b\,t^2\,XX_1^2+a\,b^2\,t^2\,XX_1^2-a^2\,b^2\,t^2\,XX_1^2\big)$$

$$\sqrt{t^2\,XX_2\,XX_3}+a^2\,(-1+b)\,c\,Q_{1,0}\,t\,\Big(4\,a\,b\,t^3-2\,\sqrt{t^2\,XX_2\,XX_3}+2\,a\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$2\,b\,\sqrt{t^2\,XX_2\,XX_3}-2\,a\,b\,\sqrt{t^2\,XX_2\,XX_3}-b\,t\,XX_1\,\sqrt{t^2\,XX_2\,XX_3}+a\,b\,t\,XX_1\,\sqrt{t^2\,XX_2\,XX_3}\Big)-$$

$$\frac{1}{8\,t\,XX_2\,XX_3}\,a\,\Big(16\,a^2\,b\,c\,t^5\,XX_2-16\,a^2\,b^2\,c\,t^5\,XX_2+16\,a^2\,b\,c\,t^5\,XX_3-16\,a^2\,b^2\,c\,t^5\,XX_3+$$

$$32\,a\,c\,t^3\,XX_2\,XX_3-16\,a^2\,c\,t^3\,XX_2\,XX_3+32\,b\,c\,t^3\,XX_2\,XX_3-80\,a\,b\,c\,t^3\,XX_2\,XX_3+$$

$$16\,a^2\,b\,c\,t^3\,XX_2\,XX_3-32\,b^2\,c\,t^3\,XX_2\,XX_3+48\,a\,b^2\,c\,t^3\,XX_2\,XX_3+32\,a^2\,b^2\,c\,t^6\,XX_2\,XX_3-$$

$$16\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+32\,a\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-16\,a^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$32\,b\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-64\,a\,b\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+32\,a^2\,b\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$16\,b^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+32\,a\,b^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-16\,a^2\,b^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$16\,a^2\,b\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+16\,a\,b^2\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$32\,a^2\,b^2\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-16\,a^2\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$32\,a\,b\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+48\,a^2\,b\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$16\,b^2\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+48\,a\,b^2\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$32\,a^2\,b^2\,c\,t^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-8\,a\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$8\,a^2\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-8\,b\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+24\,a\,b\,t\,XX_1\,XX_2\,XX_3$$

$$\sqrt{t^2\,XX_2\,XX_3}-16\,a^2\,b\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+8\,b^2\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$16\,a\,b^2\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+8\,a^2\,b^2\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$8\,c\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-8\,a\,c\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$8\,b\,c\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+8\,a\,b\,c\,t\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$8\,a^2\,b\,c\,t^4\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-8\,a\,b^2\,c\,t^4\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$16\,a^2\,b^2\,c\,t^4\,XX_1\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+2\,a\,b\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$2\,a^2\,b\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-2\,a\,b^2\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$2\,a^2\,b^2\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-2\,a\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$2\,a^2\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-2\,b\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$4\,a\,b\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-2\,a^2\,b\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$2\,b^2\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-2\,a\,b^2\,c\,t^2\,XX_1^2\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+$$

$$a\,b\,c\,t^3\,XX_1^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-a^2\,b\,c\,t^3\,XX_1^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}-$$

$$a\,b^2\,c\,t^3\,XX_1^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}+a^2\,b^2\,c\,t^3\,XX_1^3\,XX_2\,XX_3\,\sqrt{t^2\,XX_2\,XX_3}\Big)\,Q[0,0]$$

Out[ ]=

$$O[t]^{13/2}$$

```
In[•]:= (* the RHS *)
        (μ / x / Sqrt[ΔΔp] + ν Sqrt[ΔΔm] Sqrt[ΔΔ0] / x) +
          (μ0,0 / x / Sqrt[ΔΔp] + ν0,0 Sqrt[ΔΔm] Sqrt[ΔΔ0] / x) * Q[0, 0] +
          (μ0,1 / x / Sqrt[ΔΔp] + ν0,1 Sqrt[ΔΔm] Sqrt[ΔΔ0] / x) * Q0,1 +
          (μ1,0 / x / Sqrt[ΔΔp] + ν1,0 Sqrt[ΔΔm] Sqrt[ΔΔ0] / x) * Q1,0 - xposRHS1 -
          xposRHS2 - xposRHS3 - xposRHS4 - xnegRHS1 - xnegRHS2 - xnegRHS3 - xnegRHS4;
        x0RHS = Collect[%, {Q[0, 0], Q0,1, Q1,0}, Factor]
        (* check it *)
        (μ / x / Sqrt[Δp] + ν Sqrt[Δm] Sqrt[Δ0] / x) +
            (μ0,0 / x / Sqrt[Δp] + ν0,0 Sqrt[Δm] Sqrt[Δ0] / x) * Q[0, 0] +
            (μ0,1 / x / Sqrt[Δp] + ν0,1 Sqrt[Δm] Sqrt[Δ0] / x) * Q0,1 +
            (μ1,0 / x / Sqrt[Δp] + ν1,0 Sqrt[Δm] Sqrt[Δ0] / x) * Q1,0 /.
          {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0], Q0,1 → QQcxy[9, 0, 1]};
        ApplyToSeries[Coefficient[#, x, 0] &, %];
        x0RHS /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
          {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0], Q0,1 → QQcxy[9, 0, 1]};
        % - %% // Simplify
```

Out[•]=

$$
\begin{aligned}
& -\frac{1}{2}(-1+a)\, a\, b\, c\, Q_{0,1}\, t^2 \Bigl(4 a b t^2 + a XX_1 \sqrt{t^2 XX_2 XX_3} - b XX_1 \sqrt{t^2 XX_2 XX_3}\Bigr) + \\
& \frac{1}{2} a^2 (-1+b)\, c\, Q_{1,0}\, t^2 \Bigl(4 a b t^2 + a XX_1 \sqrt{t^2 XX_2 XX_3} - b XX_1 \sqrt{t^2 XX_2 XX_3}\Bigr) + \\
& \frac{1}{16 XX_2 XX_3} a \Bigl(16 a^2 b^2 t^4 XX_2 + 16 a^2 b^2 t^4 XX_3 + 16 a^2 b t^2 XX_2 XX_3 + 16 a b^2 t^2 XX_2 XX_3 - \\
& \quad 32 a^2 b^2 t^2 XX_2 XX_3 + 16 a^2 b t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 a b^2 t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 32 a^2 b^2 t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 8 a XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 8 a^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 8 b XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 a b XX_1 XX_2 XX_3 \\
& \quad \sqrt{t^2 XX_2 XX_3} - 8 a^2 b XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 8 b^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 8 a b^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 16 a^2 b^2 t^3 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 2 a^2 b t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 2 a b^2 t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 4 a^2 b^2 t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + a^2 b t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad a b^2 t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 2 a^2 b^2 t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3}\Bigr) - \\
& \frac{1}{16 XX_2 XX_3} a \Bigl(16 a^2 b^2 t^4 XX_2 + 16 a^2 b c t^4 XX_2 - 16 a b^2 c t^4 XX_2 - 16 a^2 b^2 c t^4 XX_2 + \\
& \quad 16 a^2 b^2 t^4 XX_3 + 16 a^2 b c t^4 XX_3 - 16 a b^2 c t^4 XX_3 - 16 a^2 b^2 c t^4 XX_3 + 16 a^2 b t^2 XX_2 XX_3 + \\
& \quad 16 a b^2 t^2 XX_2 XX_3 - 32 a^2 b^2 t^2 XX_2 XX_3 + 32 a c t^2 XX_2 XX_3 - 16 a^2 c t^2 XX_2 XX_3 + \\
& \quad 32 b c t^2 XX_2 XX_3 - 96 a b c t^2 XX_2 XX_3 + 16 a^2 b c t^2 XX_2 XX_3 - 48 b^2 c t^2 XX_2 XX_3 + \\
& \quad 80 a b^2 c t^2 XX_2 XX_3 + 32 a^2 b^2 c t^5 XX_2 XX_3 + 16 a^2 b t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 16 a b^2 t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 32 a^2 b^2 t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 16 a^2 c t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 a^2 b c t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 16 b^2 c t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 a b^2 c t^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 8 a XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 8 a^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 8 b XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 a b XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 8 a^2 b XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 8 b^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 8 a b^2 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 16 c XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 16 a c XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 16 b c XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 16 a b c XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 16 a^2 b^2 t^3 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 8 a^2 b c t^3 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 8 a b^2 c t^3 XX_1 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 2 a^2 b t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 2 a b^2 t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad 4 a^2 b^2 t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 2 a^2 c t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 2 a^2 b c t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + 2 b^2 c t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad 2 a b^2 c t XX_1^2 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + a^2 b t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + \\
& \quad a b^2 t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - 2 a^2 b^2 t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad a^2 c t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + a^2 b c t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} - \\
& \quad b^2 c t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3} + a b^2 c t^2 XX_1^3 XX_2 XX_3 \sqrt{t^2 XX_2 XX_3}\Bigr)\, Q[0, 0]
\end{aligned}
$$

Out[•]= $O[t]^{19/2}$

```
In[•]:= (* combining these *)
        υ0,0 = Coefficient[-x0LHS + x0RHS, Q[0, 0]];
        υ0,1 = Coefficient[-x0LHS + x0RHS, Q0,1];
        υ1,0 = Coefficient[-x0LHS + x0RHS, Q1,0];
        υ = -x0LHS + x0RHS /. {Q[0, 0] → 0, Q0,1 → 0, Q1,0 → 0};
        (* check it *)
        υ + υ0,0 * Q[0, 0] + υ0,1 * Q0,1 + υ1,0 * Q1,0 /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
         {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0], Q0,1 → QQcxy[9, 0, 1]}
Out[•]= O[t]^9

In[•]:= (* give it a name *)
        Clear[H0];
        H0[n_] := H0[n] = Simplificate /@ (ApplyToSeries[Simplify, #] & /@
              ({υ, υ0,0, υ0,1, υ1,0} /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}))

In[•]:= H0[9].{1, Q[0, 0], Q0,1, Q1,0};
        % /. {Q[0, 0] → QQcxy[9, 0, 0], Q0,1 → QQcxy[9, 0, 1], Q1,0 → QQcxy[9, 1, 0]} //
          Simplify;
        % // Simplificate
Out[•]= O[t]^9

In[•]:= (* now can this equation be combined with any of the others? *)
        Drop[#, 1] & /@ {H0[5], Hx1[5], Hx3[5]};
        ApplyToSeries[Factor, Det[%]]
        Drop[#, 1] & /@ {H0[5], Hx1[5], Hx5[5]};
        ApplyToSeries[Factor, Det[%]]
        Drop[#, 1] & /@ {H0[5], Hx3[5], Hx5[5]};
        ApplyToSeries[Factor, Det[%]]
        Drop[#, 1] & /@ {H0[3], Hx1[3], Hx7[3]};
        ApplyToSeries[Factor, Det[%]]
        Drop[#, 1] & /@ {H0[5], Hx3[5], Hx7[5]};
        ApplyToSeries[Factor, Det[%]]
        Drop[#, 1] & /@ {H0[3], Hx5[3], Hx7[3]};
        ApplyToSeries[Factor, Det[%]]
Out[•]= O[t]^11
Out[•]= O[t]^21/2
Out[•]= O[t]^11
Out[•]= -8 ((-2 + a) (-1 + a)^2 a^5 (a - b)^2 (-1 + b)^3 b^3 c^2) t^7 + O[t]^17/2
Out[•]= O[t]^11
Out[•]= -8 ((-1 + a)^3 a^6 (a - b)^2 (-1 + b)^3 b^3 c^2) t^7 + O[t]^17/2

In[•]:= (* yes -- it can be combined with {x1,x7} or {x5,x7} *)
```