Quarter-plane lattice paths with interacting boundaries: the Kreweras and reverse Kreweras models

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Some calculations accompanying the solution to **Kreweras** walks with general boundary weights (a,b,c). Symbols and equation numbers match the manuscript where possible.

Note: Many symbols are reused between this notebook and the reverse Kreweras notebook -- be sure to quit the kernel before switching to the other one, or use a different kernel for each.

(This block needs to be expanded to run some preliminary commands!)

Preliminaries

It will be useful to have some series to substitute into equations to check their validity.

```
In[•]:= (* shorthand to apply a function f to the terms of a series *)
ApplyToSeries[f_, S_] := MapAt[f /@ # &, S, 3]
```

```
In[•]:= (* mathematica sometimes has trouble when
 combining multiple series in the same variable *)
(* so here's a way of dealing with that *)
Simplificate[S_] :=
 Table[S[[1]]^n, {n, S[[-3]] / S[[-1]], S[[-3]] / S[[-1]] + (Length[S[[3]]] - 1) /
      S[[-1]], 1 / S[[-1]]}].S[[3]] + O[S[[1]]]^(S[[-2]] / S[[-1]])
```

```
In[•]:= (* this will also be useful *)
Needs["Notation`"]
```

```
In[•]:= Symbolize[ParsedBoxWrapper[SubscriptBox["_", "_"]]]
Symbolize[ParsedBoxWrapper[SubsuperscriptBox["_", "_", "_"]]]
```

```
In[•]:= (* calculate the coefficients (polynomials in a,b,c) recursively *)
(* let q[n,i,j] be the total weight of
 walks of length n ending at coordinate (i,j) *)
Clear[q]
q[0, 0, 0] = 1;
q[n_, i_, j_] := (q[n, i, j] = 0) /; (n < 0 || i < 0 || j < 0);
q[n_, i_, j_] :=
 (q[n, i, j] = Expand[q[n - 1, i - 1, j - 1] + q[n - 1, i + 1, j] + q[n - 1, i, j + 1]]) /;
  (i > 0 && j > 0)
q[n_, 0, j_] := (q[n, 0, j] = Expand[b q[n - 1, 1, j] + b q[n - 1, 0, j + 1]]) /; (j > 0)
q[n_, i_, 0] := (q[n, i, 0] = Expand[a q[n - 1, i, 1] + a q[n - 1, i + 1, 0]]) /; (i > 0)
q[n_, 0, 0] := (q[n, 0, 0] = Expand[c q[n - 1, 0, 1] + c q[n - 1, 1, 0]])
```

In[•]:=
```
(* then the generating functions *)
Clear[QQ, QQcx, QQcy, QQcxy, QQeval, QQcxeval, QQcyeval, QQdk, QQdkeval]
QQ[N_] := QQ[N] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * x^i * y^j, {n, 0, N}, {i, 0, n}, {j, 0, n}] + O[t]^(N + 1)]
(* coefficients of specific powers of x,y, or both *)
QQcx[N_, i_] := QQcx[N, i] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * y^j, {n, 0, N}, {j, 0, n}] + O[t]^(N + 1)]
QQcy[N_, j_] := QQcy[N, j] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * x^i, {n, 0, N}, {i, 0, n}] + O[t]^(N + 1)]
QQcxy[N_, i_, j_] := QQcxy[N, i, j] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n, {n, 0, N}] + O[t]^(N + 1)]
(* evaluating QQ at some other values of (x,y) *)
QQeval[N_, xx_, yy_] := QQeval[N, xx, yy] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * xx^i * yy^j, {n, 0, N}, {i, 0, n}, {j, 0, n}] + O[t]^(N + 1)]
QQcxeval[N_, i_, yy_] := QQcxeval[N, i, yy] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * yy^j, {n, 0, N}, {j, 0, n}] + O[t]^(N + 1)]
QQcyeval[N_, j_, xx_] := QQcyeval[N, j, xx] = ApplyToSeries[Expand,
   Sum[q[n, i, j] * t^n * xx^i, {n, 0, N}, {i, 0, n}] + O[t]^(N + 1)]
(* the generalised diagonal term *)
QQdk[N_, k_] := QQdk[N, k] = ApplyToSeries[Expand,
   Sum[q[n, i, i + k] * t^n * x^i, {n, 0, N}, {i, 0, n}] + O[t]^(N + 1)]
QQdkeval[N_, k_, xx_] := QQdkeval[N, k, xx] = ApplyToSeries[Expand,
   Sum[q[n, i, i + k] * t^n * xx^i, {n, 0, N}, {i, 0, n}] + O[t]^(N + 1)]
```

Section 5.1

In[•]:=
```
(* the kernel and A,B,G *)
K[x_, y_] := 1 - t (x y + 1 / x + 1 / y)
A = 1 / y
B = 1 / x
G = 0
```

Out[•]= $\frac{1}{y}$

Out[•]= $\frac{1}{x}$

Out[•]= 0

In[•]:=
```
(* the rhs of eqn (5.1) *)
mainFE = 1 / c + 1 / a (a - 1 - t a A) Q[x, 0] +
   1/b (b - 1 - t b B) Q[0, y] + (1/(a b c) (a c + b c - a b - a b c) + t G) Q[0, 0];
(* then verifying eqn (5.1) *)
mainFE - K[x, y] × Q[x, y] /. {Q[x, y] → QQ[12],
  Q[x, 0] → QQcy[12, 0], Q[0, y] → QQcx[12, 0], Q[0, 0] → QQcxy[12, 0, 0]}
```

Out[•]= $O[t]^{13}$

Section 5.2

In[•]:=
```
(* apply the kernel symmetries *)
mainFE0 = mainFE;
mainFE1 = mainFE0 /. {x → 1 / (x y)};
mainFE2 = mainFE1 /. {y → 1 / (x y)};
mainFE3 = mainFE2 /. {x → 1 / (x y)};
mainFE4 = mainFE3 /. {y → 1 / (x y)};
mainFE5 = mainFE4 /. {x → 1 / (x y)};
```

In[•]:=
```
(* the vector V from eqn (5.2) *)
(* the order is arbitrary *)
V = {Q[x, 0], Q[0, y], Q[1 / x / y, 0], Q[0, 1 / x / y], Q[0, x], Q[y, 0]};
(* then the coefficient matrix M *)
M = {Coefficient[mainFE0, V], Coefficient[mainFE1, V], Coefficient[mainFE2, V],
  Coefficient[mainFE3, V], Coefficient[mainFE4, V], Coefficient[mainFE5, V]}
```

Out[•]=
$$\left\{\left\{\frac{-1+a-\frac{a t}{y}}{a}, \frac{-1+b-\frac{b t}{x}}{b}, 0, 0, 0, 0\right\}, \left\{0, \frac{-1+b-b t x y}{b}, \frac{-1+a-\frac{a t}{y}}{a}, 0, 0, 0\right\}, \left\{0, 0, 0, \frac{-1+b-\frac{b t}{y}}{b}, 0, \frac{-1+a-a t x y}{a}\right\}, \left\{0, 0, 0, 0, \frac{-1+b-\frac{b t}{y}}{b}, \frac{-1+a-\frac{a t}{x}}{a}\right\}, \left\{0, 0, \frac{-1+a-\frac{a t}{x}}{a}, 0, \frac{-1+b-b t x y}{b}, 0\right\}, \left\{\frac{-1+a-a t x y}{a}, 0, 0, \frac{-1+b-\frac{b t}{x}}{b}, 0, 0\right\}\right\}$$

In[•]:=
```
(* write this using *)
Ap[x_, y_] := 1 / a (a - 1 - t a / y)
Bp[x_, y_] := 1/b (b - 1 - t b/x)
{{Ap[x, y], Bp[x, y], 0, 0, 0, 0}, {0, Bp[1 / x / y, y], Ap[1 / x / y, y], 0, 0, 0},
  {0, 0, 0, Bp[y, 1 / x / y], 0, Ap[y, 1 / x / y]},
  {0, 0, 0, 0, Bp[y, x], Ap[y, x]}, {0, 0, Ap[1 / x / y, x], 0, Bp[1 / x / y, x], 0},
  {Ap[x, 1 / x / y], 0, 0, Bp[x, 1 / x / y], 0, 0}} - M
```

Out[•]=
```
{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}
```

In[•]:=
```
(* the vector C is everything else, see eqn (5.2) *)
CC = {mainFE0, mainFE1, mainFE2, mainFE3, mainFE4, mainFE5} /. (# → 0 & /@ V)
```

Out[•]=
$$\left\{\frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}, \frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}, \frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}, \frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}, \frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}, \frac{1}{c}+\frac{(-a b+a c+b c-a b c)\, Q[0,0]}{a b c}\right\}$$

In[•]:=
```
(* M has rank 5 *)
MatrixRank[M]
```

Out[•]= 5

In[∘]:=
```
(* the vector N spans the nullspace of M, see eqn (5.4) *)
NullSpace[M^T];
(* clean up the denominators a bit *)
NN = -%[[1]] * (a t + y - a y) (1 - b + b t x y) (a t + x - a x) (b t + y - b y) / y // Factor
(* and see *)
NN.M // FullSimplify
```

Out[∘]=
$$\left\{(a t+x-a x)(b t+y-b y)(1-a+a t x y)(1-b+b t x y),\ -\frac{(-a t-x+a x)(-b t-x+b x)(b t+y-b y)(1-a+a t x y)}{x},\ \frac{(-a t-x+a x)(-b t-x+b x)(a t+y-a y)(1-b+b t x y)}{x},\ -(b t+x-b x)(a t+y-a y)(1-a+a t x y)(1-b+b t x y),\ \frac{(b t+x-b x)(a t+y-a y)(b t+y-b y)(1-a+a t x y)}{y},\ -\frac{(a t+x-a x)(a t+y-a y)(b t+y-b y)(1-b+b t x y)}{y}\right\}$$

Out[∘]= $\{0, 0, 0, 0, 0, 0\}$

In[∘]:=
```
(* unlike reverse Kreweras, this time N.C ≠ 0 *)
fullOSrhs = NN.CC // Collect[#, Q[0, 0], Factor] &
```

Out[∘]=
$$-\frac{a(a-b)b t^3(x-y)(-1+x^2 y)(-1+x y^2)}{c x y}+\frac{(a-b)(a b-a c-b c+a b c)t^3(x-y)(-1+x^2 y)(-1+x y^2)Q[0,0]}{c x y}$$

In[∘]:=
```
(* now we need to extract [y^0] of N.Q *)
fullOSlhs =
 NN.{Q[x, y], Q[1 / x / y, y], Q[y, 1 / x / y], Q[y, x], Q[1 / x / y, x], Q[x, 1 / x / y]}
```

Out[∘]=
$$-\frac{(a t+x-a x)(a t+y-a y)(b t+y-b y)(1-b+b t x y)Q\left[x,\frac{1}{x y}\right]}{y}+(a t+x-a x)(b t+y-b y)(1-a+a t x y)(1-b+b t x y)Q[x,y]+\frac{(b t+x-b x)(a t+y-a y)(b t+y-b y)(1-a+a t x y)Q\left[\frac{1}{x y},x\right]}{y}-\frac{(-a t-x+a x)(-b t-x+b x)(b t+y-b y)(1-a+a t x y)Q\left[\frac{1}{x y},y\right]}{x}-(b t+x-b x)(a t+y-a y)(1-a+a t x y)(1-b+b t x y)Q[y,x]+\frac{(-a t-x+a x)(-b t-x+b x)(a t+y-a y)(1-b+b t x y)Q\left[y,\frac{1}{x y}\right]}{x}$$

In[•]:=
```
(* this is not too complicated *)
fullOSlhsy0 = {0, 0, 0, 0, 0, 0};
fullOSlhsy0[[1]] =
 Coefficient[Coefficient[fullOSlhs, Q[x, y]], y, 0] * Q[x, 0] // Factor
CoefficientList[Coefficient[fullOSlhs, Q[1 / x / y, y]], y] // Factor;
fullOSlhsy0[[2]] = %[[1]] * Q^d_0[1 / x] + %[[2]] * Q^d_-1[1 / x] + %[[3]] * Q^d_-2[1 / x]
CoefficientList[Coefficient[fullOSlhs, Q[y, 1 / x / y]], y] // Factor;
fullOSlhsy0[[3]] = %[[1]] * Q^d_0[1 / x] + %[[2]] * Q^d_1[1 / x] / x + %[[3]] * Q^d_2[1 / x] / x^2
fullOSlhsy0[[4]] =
 Coefficient[Coefficient[fullOSlhs, Q[y, x]], y, 0] * Q[0, x] // Factor
Coefficient[fullOSlhs, Q[1 / x / y, x]] // Collect[#, y, Factor] &;
fullOSlhsy0[[5]] = Coefficient[%, y, 0] * Q[0, x] +
  Coefficient[%, y, 1] * Q_1,·[x] / x + Coefficient[%, y, 2] * Q_2,·[x] / x^2
Coefficient[fullOSlhs, Q[x, 1 / x / y]] // Collect[#, y, Factor] &
fullOSlhsy0[[6]] = Coefficient[%, y, 0] * Q[x, 0] +
  Coefficient[%, y, 1] * Q_·,1[x] / x + Coefficient[%, y, 2] * Q_·,2[x] / x^2
```

Out[•]= $(-1+a)\,(-1+b)\,b\,t\,(a\,t+x-a\,x)\,Q[x,0]$

Out[•]= $$\frac{(-1+a)\,b\,t\,(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,Q^{d}_{0}\left[\frac{1}{x}\right]}{x}-\frac{(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,(1-a-b+a\,b+a\,b\,t^2\,x)\,Q^{d}_{-1}\left[\frac{1}{x}\right]}{x}+a\,(-1+b)\,t\,(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,Q^{d}_{-2}\left[\frac{1}{x}\right]$$

Out[•]= $$-\frac{a\,(-1+b)\,t\,(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,Q^{d}_{0}\left[\frac{1}{x}\right]}{x}+\frac{(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,(1-a-b+a\,b+a\,b\,t^2\,x)\,Q^{d}_{1}\left[\frac{1}{x}\right]}{x^2}-\frac{(-1+a)\,b\,t\,(a\,t+x-a\,x)\,(b\,t+x-b\,x)\,Q^{d}_{2}\left[\frac{1}{x}\right]}{x^2}$$

Out[•]= $-(-1+a)\,a\,(-1+b)\,t\,(b\,t+x-b\,x)\,Q[0,x]$

Out[•]= $$t\,(b\,t+x-b\,x)\,(a-a^2+b-3\,a\,b+2\,a^2\,b+a^2\,b\,t^2\,x)\,Q[0,x]-\frac{(b\,t+x-b\,x)\,(-1+2\,a-a^2+b-2\,a\,b+a^2\,b-a^2\,t^2\,x-a\,b\,t^2\,x+2\,a^2\,b\,t^2\,x)\,Q_{1,\cdot}[x]}{x}+\frac{(-1+a)\,a\,(-1+b)\,t\,(b\,t+x-b\,x)\,Q_{2,\cdot}[x]}{x}$$

Out[•]= $$-t\,(a\,t+x-a\,x)\,(a+b-3\,a\,b-b^2+2\,a\,b^2+a\,b^2\,t^2\,x)+\frac{a\,(-1+b)\,b\,t^2\,(a\,t+x-a\,x)}{y}+(a\,t+x-a\,x)\,(-1+a+2\,b-2\,a\,b-b^2+a\,b^2-a\,b\,t^2\,x-b^2\,t^2\,x+2\,a\,b^2\,t^2\,x)\,y-(-1+a)\,(-1+b)\,b\,t\,x\,(a\,t+x-a\,x)\,y^2$$

Out[$\circ$]= $-t\,(a\,t+x-a\,x)\,\left(a+b-3\,a\,b-b^2+2\,a\,b^2+a\,b^2\,t^2\,x\right)\,Q[x,0]+\dfrac{(a\,t+x-a\,x)\,\left(-1+a+2\,b-2\,a\,b-b^2+a\,b^2-a\,b\,t^2\,x-b^2\,t^2\,x+2\,a\,b^2\,t^2\,x\right)\,Q_{\cdot,1}[x]}{x}-\dfrac{(-1+a)\,\left(-1+b\right)\,b\,t\,(a\,t+x-a\,x)\,Q_{\cdot,2}[x]}{x}$

In[$\circ$]:=
```
(* check it manually *)
fullOSlhs /. {Q[ecks_, why_] → QQeval[12, ecks, why]};
ApplyToSeries[Expand, %];
ApplyToSeries[Coefficient[#, y, 0] &, %];
Total[fullOSlhsy0] /. {Q[x, 0] → QQcy[12, 0], Q[0, x] → QQcxeval[12, 0, x],
    Q^d_0[1/x] → QQdkeval[12, 0, 1 / x], Q^d_1[1/x] → QQdkeval[12, 1, 1 / x],
    Q^d_2[1/x] → QQdkeval[12, 2, 1 / x], Q^d_-1[1/x] → QQdkeval[12, -1, 1 / x],
    Q^d_-2[1/x] → QQdkeval[12, -2, 1 / x], Q_1,.[x] → QQcxeval[12, 1, x],
    Q_2,.[x] → QQcxeval[12, 2, x], Q_.,1[x] → QQcy[12, 1], Q_.,2[x] → QQcy[12, 2]};
ApplyToSeries[Expand, %];
% - %%%
```

Out[$\circ$]= $O[t]^{13}$

In[$\circ$]:=
```
(* now want to use some boundary and diagonal
 relations to eliminate some of these terms *)
(* equation for Q.,1[x] *)
Qx1eqn =
 -Q_.,1[x] + t x Q[x, 0] + t / x Q_.,1[x] + t (b - 1) Q_1,1 - t / x Q_0,1 + t Q_.,2[x] + t (b - 1) Q_0,2
(* check it *)
% /. {Q[x, 0] → QQcy[12, 0], Q_.,1[x] → QQcy[12, 1], Q_.,2[x] → QQcy[12, 2],
  Q_1,1 → QQcxy[12, 1, 1], Q_0,2 → QQcxy[12, 0, 2], Q_0,1 → QQcxy[12, 0, 1]}
(* equation for Q1,.[x] *)
Q1xeqn =
 -Q_1,.[x] + t x Q[0, x] + t / x Q_1,.[x] + t (a - 1) Q_1,1 - t / x Q_1,0 + t Q_2,.[x] + t (a - 1) Q_2,0
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x],
  Q_1,.[x] → QQcxeval[12, 1, x], Q_2,.[x] → QQcxeval[12, 2, x],
  Q_1,1 → QQcxy[12, 1, 1], Q_2,0 → QQcxy[12, 2, 0], Q_1,0 → QQcxy[12, 1, 0]}
(* equation for Q[x,0] *)
Qx0eqn =
 -Q[x, 0] + 1 + t a Q_.,1[x] + t (c - a) Q_0,1 + t / x a Q[x, 0] + t (c - a) Q_1,0 - t / x a Q[0, 0]
(* check it *)
% /. {Q[x, 0] → QQcy[12, 0], Q_.,1[x] → QQcy[12, 1],
  Q_0,1 → QQcxy[12, 0, 1], Q_1,0 → QQcxy[12, 1, 0], Q[0, 0] → QQcxy[12, 0, 0]}
(* equation for Q[0,x] *)
Q0xeqn =
 -Q[0, x] + 1 + t b Q_1,.[x] + t (c - b) Q_1,0 + t / x b Q[0, x] + t (c - b) Q_0,1 - t / x b Q[0, 0]
(* check it *)
% /. {Q[0, x] → QQcxeval[12, 0, x], Q_1,.[x] → QQcxeval[12, 1, x],
```

```
  Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0], Q[0, 0] → QQcxy[12, 0, 0]}
(* equation for Q^d_{-1}[1/x] *)
Qdm1eqn = -Q^d_{-1}[1/x] + t / x Q^d_{-1}[1/x] + t Q^d_0[1/x] +
  t / x (a - 1) Q_{1,1} - t Q[0, 0] + t x Q^d_{-2}[1/x] + t / x (a - 1) Q_{2,0}
(* check it *)
% /. {Q^d_{-1}[1/x] → QQdkeval[12, -1, 1 / x],
  Q^d_0[1/x] → QQdkeval[12, 0, 1 / x], Q^d_{-2}[1/x] → QQdkeval[12, -2, 1 / x],
  Q[0, 0] → QQcxy[12, 0, 0], Q_{1,1} → QQcxy[12, 1, 1], Q_{2,0} → QQcxy[12, 2, 0]}
(* equation for Q^d_1[1/x] *)
Qdp1eqn =
 -Q^d_1[1/x] + t / x Q^d_1[1/x] + t Q^d_2[1/x] + t (b - 1) Q_{0,2} + t x Q^d_0[1/x] + t (b - 1) Q_{1,1} - t x Q[0, 0]
(* check it *)
% /. {Q^d_1[1/x] → QQdkeval[12, 1, 1 / x],
  Q^d_2[1/x] → QQdkeval[12, 2, 1 / x], Q^d_0[1/x] → QQdkeval[12, 0, 1 / x],
  Q_{0,2} → QQcxy[12, 0, 2], Q_{1,1} → QQcxy[12, 1, 1], Q[0, 0] → QQcxy[12, 0, 0]}
(* equation for Q^d_0[1/x] *)
Qd0eqn = -Q^d_0[1/x] + 1 + t / x Q^d_0[1/x] + t Q^d_1[1/x] + t (c - 1) Q_{0,1} + t x Q^d_{-1}[1/x] + t (c - 1) Q_{1,0}
(* check it *)
% /. {Q^d_1[1/x] → QQdkeval[12, 1, 1 / x], Q^d_{-1}[1/x] → QQdkeval[12, -1, 1 / x],
  Q^d_0[1/x] → QQdkeval[12, 0, 1 / x], Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0]}
(* equation for Q_{0,1} *)
Q01eqn = -Q_{0,1} + t b Q_{0,2} + t b Q_{1,1}
(* check it *)
% /. {Q_{0,1} → QQcxy[12, 0, 1], Q_{0,2} → QQcxy[12, 0, 2], Q_{1,1} → QQcxy[12, 1, 1]}
(* equation for Q[0,0] *)
Q00eqn = -Q[0, 0] + 1 + t c Q_{0,1} + t c Q_{1,0}
(* check it *)
% /. {Q[0, 0] → QQcxy[12, 0, 0], Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0]}
(* equation for Q_{1,0} *)
Q10eqn = -Q_{1,0} + t a Q_{1,1} + t a Q_{2,0}
(* check it *)
% /. {Q_{1,0} → QQcxy[12, 1, 0], Q_{1,1} → QQcxy[12, 1, 1], Q_{2,0} → QQcxy[12, 2, 0]}
(* equation for Q_{1,1} *)
Q11eqn = -Q_{1,1} + t Q[0, 0] + t Q_{1,2} + t Q_{2,1}
(* check it *)
% /. {Q_{1,1} → QQcxy[12, 1, 1], Q[0, 0] → QQcxy[12, 0, 0],
  Q_{1,2} → QQcxy[12, 1, 2], Q_{2,1} → QQcxy[12, 2, 1]}
(* equation for Q_{2,0} *)
```

```
Q20eqn = -Q_{2,0} + t a Q_{2,1} + t a Q_{3,0}
(* check it *)
% /. {Q_{2,0} → QQcxy[12, 2, 0], Q_{2,1} → QQcxy[12, 2, 1], Q_{3,0} → QQcxy[12, 3, 0]}
(* equation for Q_{0,2} *)
Q02eqn = -Q_{0,2} + t b Q_{0,3} + t b Q_{1,2}
(* check it *)
% /. {Q_{0,2} → QQcxy[12, 0, 2], Q_{0,3} → QQcxy[12, 0, 3], Q_{1,2} → QQcxy[12, 1, 2]}
(* equation for Q_{2,1} *)
Q21eqn = -Q_{2,1} + t Q_{1,0} + t Q_{2,2} + t Q_{3,1}
(* check it *)
% /. {Q_{2,1} → QQcxy[12, 2, 1], Q_{1,0} → QQcxy[12, 1, 0],
  Q_{2,2} → QQcxy[12, 2, 2], Q_{3,1} → QQcxy[12, 3, 1]}
(* equation for Q_{3,0} *)
Q30eqn = -Q_{3,0} + t a Q_{3,1} + t a Q_{4,0}
(* check it *)
% /. {Q_{3,0} → QQcxy[12, 3, 0], Q_{3,1} → QQcxy[12, 3, 1], Q_{4,0} → QQcxy[12, 4, 0]}
```

Out[•]= $(-1+b)\,Q_{0,2}\,t+(-1+b)\,Q_{1,1}\,t-\frac{Q_{0,1}\,t}{x}+t\,x\,Q[x,0]-Q_{\cdot,1}[x]+\frac{t\,Q_{\cdot,1}[x]}{x}+t\,Q_{\cdot,2}[x]$

Out[•]= $O[t]^{13}$

Out[•]= $(-1+a)\,Q_{1,1}\,t+(-1+a)\,Q_{2,0}\,t-\frac{Q_{1,0}\,t}{x}+t\,x\,Q[0,x]-Q_{1,\cdot}[x]+\frac{t\,Q_{1,\cdot}[x]}{x}+t\,Q_{2,\cdot}[x]$

Out[•]= $O[t]^{13}$

Out[•]= $1+(-a+c)\,Q_{0,1}\,t+(-a+c)\,Q_{1,0}\,t-\frac{a\,t\,Q[0,0]}{x}-Q[x,0]+\frac{a\,t\,Q[x,0]}{x}+a\,t\,Q_{\cdot,1}[x]$

Out[•]= $O[t]^{13}$

Out[•]= $1+(-b+c)\,Q_{0,1}\,t+(-b+c)\,Q_{1,0}\,t-\frac{b\,t\,Q[0,0]}{x}-Q[0,x]+\frac{b\,t\,Q[0,x]}{x}+b\,t\,Q_{1,\cdot}[x]$

Out[•]= $O[t]^{13}$

Out[•]= $\frac{(-1+a)\,Q_{1,1}\,t}{x}+\frac{(-1+a)\,Q_{2,0}\,t}{x}-t\,Q[0,0]+t\,Q^{d}_{0}\left[\frac{1}{x}\right]-Q^{d}_{-1}\left[\frac{1}{x}\right]+\frac{t\,Q^{d}_{-1}\left[\frac{1}{x}\right]}{x}+t\,x\,Q^{d}_{-2}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

Out[•]= $(-1+b)\,Q_{0,2}\,t+(-1+b)\,Q_{1,1}\,t-t\,x\,Q[0,0]+t\,x\,Q^{d}_{0}\left[\frac{1}{x}\right]-Q^{d}_{1}\left[\frac{1}{x}\right]+\frac{t\,Q^{d}_{1}\left[\frac{1}{x}\right]}{x}+t\,Q^{d}_{2}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

Out[•]= $1+(-1+c)\,Q_{0,1}\,t+(-1+c)\,Q_{1,0}\,t-Q^{d}_{0}\left[\frac{1}{x}\right]+\frac{t\,Q^{d}_{0}\left[\frac{1}{x}\right]}{x}+t\,Q^{d}_{1}\left[\frac{1}{x}\right]+t\,x\,Q^{d}_{-1}\left[\frac{1}{x}\right]$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{0,1}+b\,Q_{0,2}\,t+b\,Q_{1,1}\,t$

Out[•]= $O[t]^{13}$

Out[•]= $1+c\,Q_{0,1}\,t+c\,Q_{1,0}\,t-Q[0,0]$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{1,0} + a\, Q_{1,1}\, t + a\, Q_{2,0}\, t$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{1,1} + Q_{1,2}\, t + Q_{2,1}\, t + t\, Q[0, 0]$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{2,0} + a\, Q_{2,1}\, t + a\, Q_{3,0}\, t$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{0,2} + b\, Q_{0,3}\, t + b\, Q_{1,2}\, t$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{2,1} + Q_{1,0}\, t + Q_{2,2}\, t + Q_{3,1}\, t$

Out[•]= $O[t]^{13}$

Out[•]= $-Q_{3,0} + a\, Q_{3,1}\, t + a\, Q_{4,0}\, t$

Out[•]= $O[t]^{13}$

In[•]:=
```
(* we can then use all these to eliminate things
 from the [y^0] of the LHS of the full orbit sum *)
(* thus obtaining eqn (5.7) *)
Total[fullOSlhsy0];
% /. Solve[Q1xeqn == 0, Q2,.[x]][[1]];
% /. Solve[Q0xeqn == 0, Q1,.[x]][[1]];
% /. Solve[Qx1eqn == 0, Q.,2[x]][[1]];
% /. Solve[Qx0eqn == 0, Q.,1[x]][[1]];
% /. Solve[Qdm1eqn == 0, Qd-2[1/x]][[1]];
% /. Solve[Qd0eqn == 0, Qd-1[1/x]][[1]];
% /. Solve[Qdp1eqn == 0, Qd2[1/x]][[1]];
% /. Solve[Q01eqn == 0, Q0,2][[1]];
% /. Solve[Q00eqn == 0, Q0,1][[1]];
% /. Solve[Q10eqn == 0, Q1,1][[1]];
fullOSlhsy0v2 =
 Collect[%, {Q[__], Qd0[1/x], Qd1[1/x], Q1,0, Q2,0, Q0,1, Q0,2, Q1,1}, Factor]
```

Out[•]=

$$\begin{aligned}
&\frac{1}{c\,t\,x^3}\Big(-a^2 b t^3 + a^2 b^2 t^3 - a^2 t^2 x + 3 a^2 b t^2 x - 2 a^2 b^2 t^2 x + 2 a t x^2 - \\
&\quad a^2 t x^2 - b t x^2 - a b t x^2 + b^2 t x^2 - a b^2 t x^2 + a^2 b^2 t x^2 + a^2 b^2 t^4 x^2 + x^3 - \\
&\quad 2 a x^3 + a^2 x^3 - b x^3 + 2 a b x^3 - a^2 b x^3 + a^2 b t^3 x^3 + a b^2 t^3 x^3 - 2 a^2 b^2 t^3 x^3 - \\
&\quad a^2 t^2 x^4 + a b t^2 x^4 + a^2 b t^2 x^4 + b^2 t^2 x^4 - 3 a b^2 t^2 x^4 + a^2 b^2 t^2 x^4\Big) + \\
&\frac{1}{a\,b\,c\,t\,x^3}\Big(a^3 b^2 t^3 - a^3 b^3 t^3 - a^3 b^2 c t^3 + a^3 b^3 c t^3 + a^3 b t^2 x - 3 a^3 b^2 t^2 x + \\
&\quad 2 a^3 b^3 t^2 x - a^2 b c t^2 x + a b^2 c t^2 x + 2 a^3 b^2 c t^2 x - a b^3 c t^2 x + a^2 b^3 c t^2 x - \\
&\quad 2 a^3 b^3 c t^2 x - 2 a^2 b t x^2 + a^3 b t x^2 + a b^2 t x^2 + a^2 b^2 t x^2 - a b^3 t x^2 + a^2 b^3 t x^2 - \\
&\quad a^3 b^3 t x^2 + a^2 c t x^2 - a^3 c t x^2 - a^2 b c t x^2 + 2 a^3 b c t x^2 - b^2 c t x^2 + 2 a b^2 c t x^2 - \\
&\quad a^2 b^2 c t x^2 - 2 a^3 b^2 c t x^2 + b^3 c t x^2 - 2 a b^3 c t x^2 + a^2 b^3 c t x^2 + a^3 b^3 c t x^2 - \\
&\quad a^3 b^3 t^4 x^2 - a^3 b^2 c t^4 x^2 + a^2 b^3 c t^4 x^2 + a^3 b^3 c t^4 x^2 - a b x^3 + 2 a^2 b x^3 - a^3 b x^3 + \\
&\quad a b^2 x^3 - 2 a^2 b^2 x^3 + a^3 b^2 x^3 - a c x^3 + a^2 c x^3 + b c x^3 + a b c x^3 - 2 a^2 b c x^3 - \\
&\quad b^2 c x^3 - a b^2 c x^3 + 2 a^2 b^2 c x^3 + a b^3 c x^3 - a^2 b^3 c x^3 - a^3 b^2 t^3 x^3 - a^2 b^3 t^3 x^3 + \\
&\quad 2 a^3 b^3 t^3 x^3 - a^3 b c t^3 x^3 + 3 a^3 b^2 c t^3 x^3 + a b^3 c t^3 x^3 - a^2 b^3 c t^3 x^3 - \\
&\quad 2 a^3 b^3 c t^3 x^3 + a^3 b t^2 x^4 - a^2 b^2 t^2 x^4 - a^3 b^2 t^2 x^4 - a b^3 t^2 x^4 + 3 a^2 b^3 t^2 x^4 - \\
&\quad a^3 b^3 t^2 x^4 - a^3 c t^2 x^4 - 2 a^2 b c t^2 x^4 + 4 a^3 b c t^2 x^4 + 2 a b^2 c t^2 x^4 + a^2 b^2 c t^2 x^4 - \\
&\quad 4 a^3 b^2 c t^2 x^4 + b^3 c t^2 x^4 - 4 a b^3 c t^2 x^4 + 2 a^2 b^3 c t^2 x^4 + a^3 b^3 c t^2 x^4\Big)\, Q[0,0] + \\
&\frac{(b t + x - b x)\,(-a t + a^2 t + x - a x + a^2 t^2 x^2)\,(-b t + b^2 t + x - b x + b^2 t^2 x^2)\, Q[0, x]}{b\,t\,x^3} - \\
&\frac{(a t + x - a x)\,(-a t + a^2 t + x - a x + a^2 t^2 x^2)\,(-b t + b^2 t + x - b x + b^2 t^2 x^2)\, Q[x, 0]}{a\,t\,x^3} + \\
&\frac{1}{t\,x^4}\,(a t + x - a x)\,(b t + x - b x)\,\Big(-a t^2 + a b t^2 + t x + a t x - b t x - a b t x - x^2 + \\
&\quad b x^2 + a b t^3 x^2 + 2 a t^2 x^3 - 2 b t^2 x^3 - a b t^2 x^3\Big)\, Q^d_0\Big[\frac{1}{x}\Big] + \\
&\frac{(a t + x - a x)\,(b t + x - b x)\,(-a t - b t + 2 a b t + 2 x - a x - b x + 2 a b t^2 x^2)\, Q^d_1\big[\frac{1}{x}\big]}{x^3}
\end{aligned}$$

In[•]=

```
(* check it manually *)
fullOSlhs /. {Q[ecks_, why_] → QQeval[12, ecks, why]};
ApplyToSeries[Expand, %];
ApplyToSeries[Coefficient[#, y, 0] &, %];
fullOSlhsy0v2 /. {Q[x, 0] → QQcy[12, 0],
    Q[0, x] → QQcxeval[12, 0, x], Q^d_0[1/x] → QQdkeval[12, 0, 1 / x],
    Q^d_1[1/x] → QQdkeval[12, 1, 1 / x], Q[0, 0] → QQcxy[12, 0, 0]};
ApplyToSeries[Expand, %];
% - %%%
```

Out[•]= $O[t]^{12}$

In[•]:=
```
(* now for the RHS of the full orbit sum *)
(* we can do a partial fraction expansion of 1/K as per Lemma 6 *)
Δ = (1 - t / x) ^2 - 4 t^2 x;
Y0 = (1 - t / x - Sqrt[Δ]) / (2 t x);
Y1 = (1 - t / x + Sqrt[Δ]) / (2 t x);
{K[x, Y0], K[x, Y1]} // Simplify
ApplyToSeries[Expand@*PowerExpand, Series[Y0, {t, 0, 3}]]
ApplyToSeries[Expand@*PowerExpand, Series[Y1, {t, 0, 3}]]
```

Out[•]= $\{0, 0\}$

Out[•]= $t + \frac{t^2}{x} + \left(\frac{1}{x^2} + x\right) t^3 + O[t]^4$

Out[•]= $\frac{1}{x\,t} - \frac{1}{x^2} - t - \frac{t^2}{x} + \left(-\frac{1}{x^2} - x\right) t^3 + O[t]^4$

In[•]:=
```
(* and then analogously to eqn (4.21) *)
1 / K[x, y] - 1 / Sqrt[Δ] (1 / (1 - Y0 / y) + 1 / (1 - y / Y1) - 1) // Simplify
```

Out[•]= 0

In[•]:=
```
(* keeping these unevaluated will make calculations a bit easier *)
YY0 = (1 - t / x - Sqrt[ΔΔ]) / (2 t x);
YY1 = (1 - t / x + Sqrt[ΔΔ]) / (2 t x);
```

In[•]:=
```
(* so we can compute the [y^0] of the RHS of the full orbit sum *)
Coefficient[Expand[fullOSrhs], y, -1] / YY1 / Sqrt[ΔΔ] +
  Coefficient[Expand[fullOSrhs], y, 0] / Sqrt[ΔΔ] +
  Coefficient[Expand[fullOSrhs], y, 1] * YY0 / Sqrt[ΔΔ] +
  Coefficient[Expand[fullOSrhs], y, 2] * YY0^2 / Sqrt[ΔΔ] +
  Coefficient[Expand[fullOSrhs], y, 3] * YY0^3 / Sqrt[ΔΔ];
fullOSrhsy0 = Collect[%, Q[__], Simplify]
```

Out[•]=
$$-\frac{1}{8\,c\,x^4\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}}\,a\,(a-b)\,b\,\Big(3\,t^4\left(-1+2\,x^3+8\,x^6\right)+x^4\left(-1+\sqrt{\Delta\Delta}\right)^3\left(1+\sqrt{\Delta\Delta}\right)+2\,t^2\,x^2\left(-1+\sqrt{\Delta\Delta}\right)\left(6+x^3\left(3+\sqrt{\Delta\Delta}\right)+\sqrt{\Delta\Delta}\right)+2\,t\,x^3\left(-1+\sqrt{\Delta\Delta}\right)^2\left(3+x^3\left(1+\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)-2\,t^3\,x\left(-5+4\,x^6\left(1+\sqrt{\Delta\Delta}\right)+x^3\left(1+5\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)\Big)+$$
$$\frac{1}{8\,c\,x^4\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}}\,(a-b)\,\left(-b\,c+a\,(b-c+b\,c)\right)\Big(3\,t^4\left(-1+2\,x^3+8\,x^6\right)+x^4\left(-1+\sqrt{\Delta\Delta}\right)^3\left(1+\sqrt{\Delta\Delta}\right)+2\,t^2\,x^2\left(-1+\sqrt{\Delta\Delta}\right)\left(6+x^3\left(3+\sqrt{\Delta\Delta}\right)+\sqrt{\Delta\Delta}\right)+2\,t\,x^3\left(-1+\sqrt{\Delta\Delta}\right)^2\left(3+x^3\left(1+\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)-2\,t^3\,x\left(-5+4\,x^6\left(1+\sqrt{\Delta\Delta}\right)+x^3\left(1+5\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)\Big)\,Q[0,0]$$

In[•]:=
```
(* we can check it *)
fullOSrhs / K[x, y] /. Q[0, 0] → QQcxy[12, 0, 0];
ApplyToSeries[Expand, %];
ApplyToSeries[Select[# + y^π + y^(2 π), Exponent[#, y] == 0 &] &, %];
fullOSrhsy0 /. ΔΔ → Δ /. Q[0, 0] → QQcxy[12, 0, 0];
ApplyToSeries[Expand, %];
% - %%%
```

Out[•]= $O[t]^{16}$

In[•]:=
```
(* and check it some more *)
fullOSlhsy0v2 - fullOSrhsy0;
% /. ΔΔ → Δ /. {Q[ecks_, why_] → QQeval[12, ecks, why],
   Q0d[1/x] → QQdkeval[12, 0, 1/x], Q1d[1/x] → QQdkeval[12, 1, 1/x]}
```

Out[•]= $O[t]^{12}$

In[•]:=
```
(* now to compute the [x^>] part of this *)
(* unlike reverse Kreweras, we will not need the [x^<] part *)
(* however, we unfortunately end up
 leaving the realm of algebraic functions here *)
```

In[•]:=
```
(* LHS is straightforward *)
(*eqn (5.11)*)
fullOSlhsy0xpos = {0, 0, 0, 0, 0, 0};
(fullOSlhsy0v2 /. {Q[__] → 0, Q0d[1/x] → 0, Q1d[1/x] → 0}) // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[1]] = Select[%, Exponent[#, x] > 0 &]
Coefficient[fullOSlhsy0v2, Q[0, 0]] // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[2]] = Select[%, Exponent[#, x] > 0 &] * Q[0, 0]
Coefficient[fullOSlhsy0v2, Q[0, x]] // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[3]] = Select[%, Exponent[#, x] > 0 &] * Q[0, x] +
   Select[%, Exponent[#, x] == 0 &] * (Q[0, x] - Q[0, 0]) +
   Select[%, Exponent[#, x] == -1 &] * (Q[0, x] - Q[0, 0] - x Q0,1) +
   Select[%, Exponent[#, x] == -2 &] * (Q[0, x] - Q[0, 0] - x Q0,1 - x^2 Q0,2) +
   Select[%, Exponent[#, x] == -3 &] * (Q[0, x] - Q[0, 0] - x Q0,1 - x^2 Q0,2 - x^3 Q0,3)
Coefficient[fullOSlhsy0v2, Q[x, 0]] // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[4]] = Select[%, Exponent[#, x] > 0 &] * Q[x, 0] +
   Select[%, Exponent[#, x] == 0 &] * (Q[x, 0] - Q[0, 0]) +
   Select[%, Exponent[#, x] == -1 &] * (Q[x, 0] - Q[0, 0] - x Q1,0) +
   Select[%, Exponent[#, x] == -2 &] * (Q[x, 0] - Q[0, 0] - x Q1,0 - x^2 Q2,0) +
   Select[%, Exponent[#, x] == -3 &] * (Q[x, 0] - Q[0, 0] - x Q1,0 - x^2 Q2,0 - x^3 Q3,0)
Coefficient[fullOSlhsy0v2, Q0d[1/x]] // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[5]] = Select[%, Exponent[#, x] == 1 &] * Q[0, 0]
Coefficient[fullOSlhsy0v2, Q1d[1/x]] // Collect[#, x, Factor] &;
fullOSlhsy0xpos[[6]] = Select[%, Exponent[#, x] == 1 &] * Q0,1
```

Out[•]= $$\frac{\left(-a^2+a\,b+a^2\,b+b^2-3\,a\,b^2+a^2\,b^2\right)\,t\,x}{c}$$

Out[•]= $$\frac{1}{a\,b\,c}\left(a^3\,b-a^2\,b^2-a^3\,b^2-a\,b^3+3\,a^2\,b^3-a^3\,b^3-a^3\,c-2\,a^2\,b\,c+4\,a^3\,b\,c+2\,a\,b^2\,c+a^2\,b^2\,c-4\,a^3\,b^2\,c+b^3\,c-4\,a\,b^3\,c+2\,a^2\,b^3\,c+a^3\,b^3\,c\right)\,t\,x\,Q[0,0]$$

Out[•]= $$\begin{aligned}&\left(\frac{t\left(a^2-2\,a^2\,b+b^2-a\,b^2+a^2\,b^2-b^3+a\,b^3+a^2\,b^3\,t^3\right)\,x}{b}-a^2\left(-1+b\right)\,b\,t^3\,x^2\right)Q[0,x]-\\&\frac{\left(-1+a+2\,b-2\,a\,b-b^2+a\,b^2+a\,b^2\,t^3-2\,a^2\,b^2\,t^3-b^3\,t^3+2\,a^2\,b^3\,t^3\right)\left(-Q[0,0]+Q[0,x]\right)}{b\,t}+\\&\frac{1}{b\,x}\left(-a+a^2+2\,a\,b-2\,a^2\,b+b^2-2\,a\,b^2+a^2\,b^2-b^3+a\,b^3-a^2\,b^2\,t^3-a\,b^3\,t^3+2\,a^2\,b^3\,t^3\right)\\&\left(-Q_{0,1}\,x-Q[0,0]+Q[0,x]\right)-\\&\frac{(-1+a)\,(1+a)\,\left(-1+b\right)\,b\,t\left(-Q_{0,1}\,x-Q_{0,2}\,x^2-Q[0,0]+Q[0,x]\right)}{x^2}+\\&\frac{(-1+a)\,a\left(-1+b\right)\,b\,t^2\left(-Q_{0,1}\,x-Q_{0,2}\,x^2-Q_{0,3}\,x^3-Q[0,0]+Q[0,x]\right)}{x^3}\end{aligned}$$

Out[•]= $$\begin{aligned}&\left(-\frac{t\left(a^2-a^3-a^2\,b+a^3\,b+b^2-2\,a\,b^2+a^2\,b^2+a^3\,b^2\,t^3\right)\,x}{a}+(-1+a)\,a\,b^2\,t^3\,x^2\right)Q[x,0]+\\&\frac{\left(-1+2\,a-a^2+b-2\,a\,b+a^2\,b-a^3\,t^3+a^2\,b\,t^3-2\,a^2\,b^2\,t^3+2\,a^3\,b^2\,t^3\right)\left(-Q[0,0]+Q[x,0]\right)}{a\,t}-\\&\frac{1}{a\,x}\left(a^2-a^3-b+2\,a\,b-2\,a^2\,b+a^3\,b+b^2-2\,a\,b^2+a^2\,b^2-a^3\,b\,t^3-a^2\,b^2\,t^3+2\,a^3\,b^2\,t^3\right)\\&\left(-Q_{1,0}\,x-Q[0,0]+Q[x,0]\right)+\\&\frac{(-1+a)\,a\left(-1+b\right)\left(1+b\right)\,t\left(-Q_{1,0}\,x-Q_{2,0}\,x^2-Q[0,0]+Q[x,0]\right)}{x^2}-\\&\frac{(-1+a)\,a\left(-1+b\right)\,b\,t^2\left(-Q_{1,0}\,x-Q_{2,0}\,x^2-Q_{3,0}\,x^3-Q[0,0]+Q[x,0]\right)}{x^3}\end{aligned}$$

Out[•]= $$-(-1+a)\left(-1+b\right)\left(-2\,a+2\,b+a\,b\right)\,t\,x\,Q[0,0]$$

Out[•]= $$2\,(-1+a)\,a\left(-1+b\right)\,b\,Q_{0,1}\,t^2\,x$$

In[•]:=
```
(* check it *)
fullOSlhsy0v2 /.
   {Q[0, 0] → QQcxy[12, 0, 0], Q[x, 0] → QQcy[12, 0], Q[0, x] → QQcxeval[12, 0, x],
    Q_0^d[1/x] → QQdkeval[12, 0, 1 / x], Q_1^d[1/x] → QQdkeval[12, 1, 1 / x]};
ApplyToSeries[Select[Expand[#] + x ^ (-π) + x ^ (-2 π), Exponent[#, x] > 0 &] &, %];
Total[fullOSlhsy0xpos] /.
   {Q[0, 0] → QQcxy[12, 0, 0], Q[x, 0] → QQcy[12, 0], Q[0, x] → QQcxeval[12, 0, x],
    Q_{0,1} → QQcxy[12, 0, 1], Q_{1,0} → QQcxy[12, 1, 0], Q_{0,3} → QQcxy[12, 0, 3],
    Q_{0,2} → QQcxy[12, 0, 2], Q_{2,0} → QQcxy[12, 2, 0], Q_{3,0} → QQcxy[12, 3, 0]};
ApplyToSeries[Expand, %];
% - %%%
```

Out[•]= $O[t]^{12}$

In[∘]:=
```
(* can do some eliminations *)
Total[fullOSlhsy0xpos] /. Solve[Q02eqn == 0, Q0,3][[1]];
% /. Solve[Q01eqn == 0, Q0,2][[1]];
% /. Solve[Q11eqn == 0, Q1,2][[1]];
% /. Solve[Q00eqn == 0, Q0,1][[1]];
% /. Solve[Q10eqn == 0, Q1,1][[1]];
% /. Solve[Q20eqn == 0, Q2,1][[1]];
fullOSlhsy0xposv2 = Collect[%, {Q[__], Q1,0, Q2,0, Q3,0}, Factor]
(* check it *)
Total[fullOSlhsy0xpos] /.
  {Q[0, 0] → QQcxy[12, 0, 0], Q[x, 0] → QQcy[12, 0], Q[0, x] → QQcxeval[12, 0, x],
   Q0,1 → QQcxy[12, 0, 1], Q1,0 → QQcxy[12, 1, 0], Q0,3 → QQcxy[12, 0, 3],
   Q0,2 → QQcxy[12, 0, 2], Q2,0 → QQcxy[12, 2, 0], Q3,0 → QQcxy[12, 3, 0]};
ApplyToSeries[Expand, %];
fullOSlhsy0xposv2 /.
  {Q[0, 0] → QQcxy[12, 0, 0], Q[x, 0] → QQcy[12, 0], Q[0, x] → QQcxeval[12, 0, x],
   Q1,0 → QQcxy[12, 1, 0], Q2,0 → QQcxy[12, 2, 0], Q3,0 → QQcxy[12, 3, 0]};
ApplyToSeries[Expand, %];
% - %%%
```

Out[∘]=

$$-(-1+a)\,a\,(-1+b)\,(2+b)\,Q_{2,0}\,t+2\,(-1+a)\,a\,(-1+b)\,b\,Q_{3,0}\,t^2+$$
$$\frac{1}{c\,t\,x^2}\big(a\,b\,t^2-a^2\,b\,t^2-a\,b^2\,t^2+a^2\,b^2\,t^2+a\,t\,x-a^2\,t\,x-b\,t\,x-a\,b\,t\,x+2\,a^2\,b\,t\,x+b^2\,t\,x-a^2\,b^2\,t\,x-x^2+a\,x^2+2\,b\,x^2-2\,a\,b\,x^2-b^2\,x^2+a\,b^2\,x^2-a^2\,b\,t^3\,x^2-a\,b^2\,t^3\,x^2+2\,a^2\,b^2\,t^3\,x^2-a^2\,t^2\,x^3-a\,b\,t^2\,x^3+3\,a^2\,b\,t^2\,x^3+b^2\,t^2\,x^3-a\,b^2\,t^2\,x^3-a^2\,b^2\,t^2\,x^3\big)+$$
$$\frac{1}{a\,x^2}\,Q_{1,0}\,\big(2\,a^2\,b\,t^2-2\,a^3\,b\,t^2-2\,a^2\,b^2\,t^2+2\,a^3\,b^2\,t^2-2\,a^2\,b\,t\,x+2\,a^3\,b\,t\,x+2\,a^2\,b^2\,t\,x-2\,a^3\,b^2\,t\,x+a^2\,x^2-a^3\,x^2-2\,b\,x^2+4\,a\,b\,x^2-3\,a^2\,b\,x^2+a^3\,b\,x^2+2\,b^2\,x^2-4\,a\,b^2\,x^2+2\,a^2\,b^2\,x^2-2\,a^3\,b\,t^3\,x^2-2\,a^2\,b^2\,t^3\,x^2+4\,a^3\,b^2\,t^3\,x^2-2\,a^2\,b\,t^2\,x^3+2\,a^3\,b\,t^2\,x^3+2\,a^2\,b^2\,t^2\,x^3-2\,a^3\,b^2\,t^2\,x^3\big)-$$
$$\frac{1}{a\,b\,c\,t\,x^2}\big(a^2\,b^2\,t^2-a^3\,b^2\,t^2-a^2\,b^3\,t^2+a^3\,b^3\,t^2+a^2\,b\,c\,t^2-a^3\,b\,c\,t^2-a\,b^2\,c\,t^2+a^3\,b^2\,c\,t^2+a\,b^3\,c\,t^2-a^2\,b^3\,c\,t^2+a^2\,b\,t\,x-a^3\,b\,t\,x-a\,b^2\,t\,x-a^2\,b^2\,t\,x+2\,a^3\,b^2\,t\,x+a\,b^3\,t\,x-a^3\,b^3\,t\,x-a^2\,c\,t\,x+a^3\,c\,t\,x+a^2\,b\,c\,t\,x-a^3\,b\,c\,t\,x+b^2\,c\,t\,x-a\,b^2\,c\,t\,x-b^3\,c\,t\,x+a\,b^3\,c\,t\,x-a\,b\,x^2+a^2\,b\,x^2+2\,a\,b^2\,x^2-2\,a^2\,b^2\,x^2-a\,b^3\,x^2+a^2\,b^3\,x^2+a\,c\,x^2-a^2\,c\,x^2-b\,c\,x^2+a^2\,b\,c\,x^2+b^2\,c\,x^2-a\,b^2\,c\,x^2-a^3\,b^2\,t^3\,x^2-a^2\,b^3\,t^3\,x^2+2\,a^3\,b^3\,t^3\,x^2-a^3\,b\,c\,t^3\,x^2+a^2\,b^2\,c\,t^3\,x^2+a^3\,b^2\,c\,t^3\,x^2+a\,b^3\,c\,t^3\,x^2-3\,a^2\,b^3\,c\,t^3\,x^2+a^3\,b^3\,c\,t^3\,x^2-a^3\,b\,t^2\,x^3-a^2\,b^2\,t^2\,x^3+3\,a^3\,b^2\,t^2\,x^3+a\,b^3\,t^2\,x^3-a^2\,b^3\,t^2\,x^3-a^3\,b^3\,t^2\,x^3+a^3\,c\,t^2\,x^3-2\,a^3\,b\,c\,t^2\,x^3+a^3\,b^2\,c\,t^2\,x^3-b^3\,c\,t^2\,x^3+2\,a\,b^3\,c\,t^2\,x^3-a^2\,b^3\,c\,t^2\,x^3\big)\,Q[0,0]+$$
$$\frac{(b\,t+x-b\,x)\,(-a\,t+a^2\,t+x-a\,x+a^2\,t^2\,x^2)\,(-b\,t+b^2\,t+x-b\,x+b^2\,t^2\,x^2)\,Q[0,x]}{b\,t\,x^3}-$$
$$\frac{(a\,t+x-a\,x)\,(-a\,t+a^2\,t+x-a\,x+a^2\,t^2\,x^2)\,(-b\,t+b^2\,t+x-b\,x+b^2\,t^2\,x^2)\,Q[x,0]}{a\,t\,x^3}$$

Out[∘]= $O[t]^{12}$

In[•]:=
```
(* the [x^>] part of the RHS is probably not algebraic *)
(* but it will be useful to name the coefficients *)
η = fullOSrhsy0 /. Q[0, 0] → 0
η0,0 = Coefficient[fullOSrhsy0, Q[0, 0]]
(* and then *)
(* this is eqn (5.12) *)
fullOSrhsy0xpos = θ + θ0,0 Q[0, 0]
```

Out[•]=
$$-\frac{1}{8\,c\,x^4\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}}\,a\,(a-b)\,b\left(3\,t^4\left(-1+2\,x^3+8\,x^6\right)+x^4\left(-1+\sqrt{\Delta\Delta}\right)^3\left(1+\sqrt{\Delta\Delta}\right)+2\,t^2\,x^2\left(-1+\sqrt{\Delta\Delta}\right)\left(6+x^3\left(3+\sqrt{\Delta\Delta}\right)+\sqrt{\Delta\Delta}\right)+2\,t\,x^3\left(-1+\sqrt{\Delta\Delta}\right)^2\left(3+x^3\left(1+\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)-2\,t^3\,x\left(-5+4\,x^6\left(1+\sqrt{\Delta\Delta}\right)+x^3\left(1+5\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)\right)$$

Out[•]=
$$\frac{1}{8\,c\,x^4\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}}\,(a-b)\,(-b\,c+a\,(b-c+b\,c))\left(3\,t^4\left(-1+2\,x^3+8\,x^6\right)+x^4\left(-1+\sqrt{\Delta\Delta}\right)^3\left(1+\sqrt{\Delta\Delta}\right)+2\,t^2\,x^2\left(-1+\sqrt{\Delta\Delta}\right)\left(6+x^3\left(3+\sqrt{\Delta\Delta}\right)+\sqrt{\Delta\Delta}\right)+2\,t\,x^3\left(-1+\sqrt{\Delta\Delta}\right)^2\left(3+x^3\left(1+\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)-2\,t^3\,x\left(-5+4\,x^6\left(1+\sqrt{\Delta\Delta}\right)+x^3\left(1+5\sqrt{\Delta\Delta}\right)+2\sqrt{\Delta\Delta}\right)\right)$$

Out[•]= $\theta+\theta_{0,0}\,Q[0,0]$

In[•]:=
```
(* and we can evaluate them manually *)
Clear[θs, θs0,0]
θs[N_] :=
 θs[N] = ApplyToSeries[Select[Expand[#] + x ^ (-π) + x ^ (-2 π), Exponent[#, x] > 0 &] &,
   Series[η /. ΔΔ → Δ, {t, 0, N}]]
θs0,0[N_] := θs0,0[N] = ApplyToSeries[
   Select[Expand[#] + x ^ (-π) + x ^ (-2 π), Exponent[#, x] > 0 &] &,
   Series[η0,0 /. ΔΔ → Δ, {t, 0, N}]]
```

In[•]:=
```
ApplyToSeries[Factor, θs[9]]
ApplyToSeries[Factor, θs0,0[9]]
```

Out[•]=
$$\frac{a\,(a-b)\,b\,x^2\,t^3}{c}+\frac{a\,(a-b)\,b\,x^3\,t^5}{c}+\frac{a\,(a-b)\,b\,x^2\,t^6}{c}+\frac{2\,a\,(a-b)\,b\,x^4\,t^7}{c}+\frac{5\,a\,(a-b)\,b\,x^3\,t^8}{c}+\frac{5\,a\,(a-b)\,b\,x^2\,(1+x)\left(1-x+x^2\right)t^9}{c}+O[t]^{10}$$

Out[•]=
$$-\frac{(a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^2\,t^3}{c}-\frac{(a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^3\,t^5}{c}-\frac{(a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^2\,t^6}{c}-\frac{2\,((a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^4)\,t^7}{c}-\frac{5\,((a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^3)\,t^8}{c}-\frac{5\,((a-b)\,(a\,b-a\,c-b\,c+a\,b\,c)\,x^2\,(1+x)\left(1-x+x^2\right))\,t^9}{c}+O[t]^{10}$$

Section 5.3

In[∘]:=
```
(* the vector V2 from eqn (5.14) *)
V2 = {Q[0, y], Q[1 / x / y, 0], Q[0, 1 / x / y], Q[y, 0]};
(* then the coefficient matrix M2 *)
M2 = {Coefficient[mainFE0, V2], Coefficient[mainFE1, V2],
  Coefficient[mainFE2, V2], Coefficient[mainFE3, V2],
  Coefficient[mainFE4, V2], Coefficient[mainFE5, V2]}
```

Out[∘]=
$$\left\{\left\{\frac{-1+b-\frac{b\,t}{x}}{b},0,0,0\right\},\left\{\frac{-1+b-b\,t\,x\,y}{b},\frac{-1+a-\frac{a\,t}{y}}{a},0,0\right\},\left\{0,0,\frac{-1+b-\frac{b\,t}{y}}{b},\frac{-1+a-a\,t\,x\,y}{a}\right\},\left\{0,0,0,\frac{-1+a-\frac{a\,t}{x}}{a}\right\},\left\{0,\frac{-1+a-\frac{a\,t}{x}}{a},0,0\right\},\left\{0,0,\frac{-1+b-\frac{b\,t}{x}}{b},0\right\}\right\}$$

In[∘]:=
```
(* the vector C2 is everything else, see eqn (5.14) *)
CC2 = {mainFE0, mainFE1, mainFE2, mainFE3, mainFE4, mainFE5} /.
  {Q[0, y] → 0, Q[1 / x / y, 0] → 0, Q[0, 1 / x / y] → 0, Q[y, 0] → 0}
```

Out[∘]=
$$\left\{\frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c}+\frac{\left(-1+a-\frac{a\,t}{y}\right)Q[x,0]}{a},\right.$$
$$\frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c},\ \frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c},$$
$$\frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c}+\frac{\left(-1+b-\frac{b\,t}{y}\right)Q[0,x]}{b},$$
$$\frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c}+\frac{(-1+b-b\,t\,x\,y)\,Q[0,x]}{b},$$
$$\left.\frac{1}{c}+\frac{(-a\,b+a\,c+b\,c-a\,b\,c)\,Q[0,0]}{a\,b\,c}+\frac{(-1+a-a\,t\,x\,y)\,Q[x,0]}{a}\right\}$$

In[•]:=
```
(* M2 has rank 4 *)
MatrixRank[M2]
(* so we have two choices for the nullspace vector N2 *)
NullSpace[(M2)ᵀ]
(* choose this one, see eqn (5.16) *)
NN2 = Select[%, Last[#] == 0 &][[1]] * -(-b t - x + b x) (a t + y - a y) / y // Factor
(* check *)
NN2.M2 // Simplify
```

Out[•]= 4

Out[•]= $\left\{\left\{0, 0, \frac{(-b t - x + b x)\, y}{x\,(b t + y - b y)}, \frac{(b t + x - b x)\, y\,(1 - a + a t x y)}{(a t + x - a x)\,(b t + y - b y)}, 0, 1\right\}, \left\{-\frac{(a t + x - a x)\, y\,(1 - b + b t x y)}{(-b t - x + b x)\,(a t + y - a y)}, \frac{(-a t - x + a x)\, y}{x\,(a t + y - a y)}, 0, 0, 1, 0\right\}\right\}$

Out[•]= $\left\{(a t + x - a x)\,(1 - b + b t x y), -\frac{(-a t - x + a x)\,(-b t - x + b x)}{x}, 0, 0, \frac{(b t + x - b x)\,(a t + y - a y)}{y}, 0\right\}$

Out[•]= {0, 0, 0, 0}

In[•]:=
```
(* this time we divide by the kernel and take the y^0 term,
as per eqn (5.17) *)
```

In[•]:=
```
(* the LHS is straightforward *)
halfOSlhs =
 NN2.{Q[x, y], Q[1 / x / y, y], Q[y, 1 / x / y], Q[y, x], Q[1 / x / y, x], Q[x, 1 / x / y]}
halfOSlhsy0 = {0, 0, 0};
Coefficient[halfOSlhs, Q[x, y]] // Collect[#, y, Factor] &;
halfOSlhsy0[[1]] = Coefficient[%, y, 0] * Q[x, 0]
Coefficient[halfOSlhs, Q[1 / x / y, x]] // Collect[#, y, Factor] &
halfOSlhsy0[[2]] = Coefficient[%, y, 0] * Q[0, x]
Coefficient[halfOSlhs, Q[1 / x / y, y]] // Collect[#, y, Factor] &
halfOSlhsy0[[3]] = % * Q0d[1 / x]
```

Out[•]= $(a t + x - a x)\,(1 - b + b t x y)\, Q[x, y] + \frac{(b t + x - b x)\,(a t + y - a y)\, Q\left[\frac{1}{x y}, x\right]}{y} - \frac{(-a t - x + a x)\,(-b t - x + b x)\, Q\left[\frac{1}{x y}, y\right]}{x}$

Out[•]= $-(-1 + b)\,(a t + x - a x)\, Q[x, 0]$

Out[•]= $-(-1 + a)\,(b t + x - b x) + \frac{a t\,(b t + x - b x)}{y}$

Out[•]= $-(-1 + a)\,(b t + x - b x)\, Q[0, x]$

Out[•]= $-\frac{(-a t - x + a x)\,(-b t - x + b x)}{x}$

Out[•]= $-\frac{(-a t - x + a x)\,(-b t - x + b x)\, Q^{d}_{0}\left[\frac{1}{x}\right]}{x}$

In[•]:=
```
(* check it *)
halfOSlhs /. {Q[ecks_, why_] → QQeval[12, ecks, why]};
ApplyToSeries[Coefficient[Expand[#], y, 0] &, %];
Total[halfOSlhsy0] /. {Q[x, 0] → QQcy[12, 0],
    Q[0, x] → QQcxeval[12, 0, x], Q^d_0[1/x] → QQdkeval[12, 0, 1 / x]};
% - %% // Simplify
```

Out[•]= $O[t]^{13}$

In[•]:=
```
(* as for the RHS *)
(* this will be divided by the kernel *)
halfOSrhs = NN₂.CC₂ // Collect[#, Q[__], Collect[#, y, Factor] &] &
```

Out[•]=
$$-\frac{a\,b\,t^2 - x^2 + a\,x^2 + b\,x^2 - a\,b\,x^2}{c\,x} + \frac{a\,t\,(b\,t + x - b\,x)}{c\,y} + \frac{b\,t\,x\,(a\,t + x - a\,x)\,y}{c} + \left(\frac{(a\,b - a\,c - b\,c + a\,b\,c)\,(a\,b\,t^2 - x^2 + a\,x^2 + b\,x^2 - a\,b\,x^2)}{a\,b\,c\,x} - \frac{(a\,b - a\,c - b\,c + a\,b\,c)\,t\,(b\,t + x - b\,x)}{b\,c\,y} - \frac{(a\,b - a\,c - b\,c + a\,b\,c)\,t\,x\,(a\,t + x - a\,x)\,y}{a\,c}\right) Q[0, 0] + \left(-\frac{(b\,t + x - b\,x)\,(1 - a - b + a\,b + a\,b\,t^2\,x)}{b} + \frac{a\,(-1 + b)\,t\,(b\,t + x - b\,x)}{b\,y} + (-1 + a)\,t\,x\,(b\,t + x - b\,x)\,y\right) Q[0, x] + \left(-\frac{(a\,t + x - a\,x)\,(1 - a - b + a\,b + a\,b\,t^2\,x)}{a} + \frac{(-1 + b)\,t\,(a\,t + x - a\,x)}{y} + \frac{(-1 + a)\,b\,t\,x\,(a\,t + x - a\,x)\,y}{a}\right) Q[x, 0]$$

In[•]:=
```
(* we've already seen the factorisation of the kernel,
so we know how to deal with this *)
(* so now we can compute the y^0 term of the RHS *)
Coefficient[halfOSrhs, y, -1] / YY1 / Sqrt[ΔΔ] +
  Coefficient[halfOSrhs, y, 0] / Sqrt[ΔΔ] +
  Coefficient[halfOSrhs, y, 1] * YY0 / Sqrt[ΔΔ];
halfOSrhsy0 = Collect[%, Q[__], Simplify]
```

Out[•]=

$$\Big(a\left(2x^2\left(t+2t^2x^2-x\left(1+\sqrt{\Delta\Delta}\right)\right)+b\,(t-x)\left(t^2\left(3+4x^3\right)-2tx\left(1+\sqrt{\Delta\Delta}\right)-x^2\left(1+\sqrt{\Delta\Delta}\right)^2\right)+x\left(t-x\left(1+\sqrt{\Delta\Delta}\right)\right)\left(-2x+b\left(t+x+x\sqrt{\Delta\Delta}\right)\right)\right)\Big)\Big/\left(2cx\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}\right)-\frac{1}{2abcx\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}}\left(-bc+a\left(b-c+bc\right)\right)\left(a\left(2x^2\left(t+2t^2x^2-x\left(1+\sqrt{\Delta\Delta}\right)\right)+b\,(t-x)\left(t^2\left(3+4x^3\right)-2tx\left(1+\sqrt{\Delta\Delta}\right)-x^2\left(1+\sqrt{\Delta\Delta}\right)^2\right)\right)+x\left(t-x\left(1+\sqrt{\Delta\Delta}\right)\right)\left(-2x+b\left(t+x+x\sqrt{\Delta\Delta}\right)\right)\right)Q[0,0]+\Big(\left(b\,(t-x)+x\right)\left(-2x\left((-1+a)\,t+2at^2x^2-(-1+a)\,x\left(1+\sqrt{\Delta\Delta}\right)\right)+b\left(2at^3x^2+t^2\left(-1+a-2ax^3\left(-1+\sqrt{\Delta\Delta}\right)\right)-(-1+a)\,x^2\left(1+\sqrt{\Delta\Delta}\right)^2\right)\right)Q[0,x]\Big)\Big/\left(2bx\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}\right)+\Big(\left(a\,(t-x)+x\right)\left(-2x\left((-1+a)\,t+2at^2x^2-(-1+a)\,x\left(1+\sqrt{\Delta\Delta}\right)\right)+b\left(2at^3x^2+t^2\left(-1+a-2ax^3\left(-1+\sqrt{\Delta\Delta}\right)\right)-(-1+a)\,x^2\left(1+\sqrt{\Delta\Delta}\right)^2\right)\right)Q[x,0]\Big)\Big/\left(2ax\left(-t+x+x\sqrt{\Delta\Delta}\right)\sqrt{\Delta\Delta}\right)$$

In[•]:=
```
(* check it *)
halfOSrhs / K[x, y] /.
  {Q[0, 0] → QQcxy[12, 0, 0], Q[0, x] → QQcxeval[12, 0, x], Q[x, 0] → QQcy[12, 0]};
ApplyToSeries[Coefficient[Expand[#], y, 0] &, %];
halfOSrhsy0 /. ΔΔ → Δ /.
  {Q[0, 0] → QQcxy[12, 0, 0], Q[0, x] → QQcxeval[12, 0, x], Q[x, 0] → QQcy[12, 0]};
ApplyToSeries[Expand@*Simplify, %];
% - %%%
```

Out[•]= $O[t]^{13}$

In[•]:=
```
(* and check some more *)
Total[halfOSlhsy0] - halfOSrhsy0 /. ΔΔ → Δ /.
  {Q[0, 0] → QQcxy[12, 0, 0], Q[0, x] → QQcxeval[12, 0, x],
   Q[x, 0] → QQcy[12, 0], Q0d[1/x] → QQdkeval[12, 0, 1 / x]};
ApplyToSeries[Simplify,
 %]
```

Out[•]= $O[t]^{13}$

In[•]:=
```
(* now combining the full-
 orbit sum with the half-orbit sum to obtain eqn (5.20) *)
(* eliminating Q[0,x] *)
Total[halfOSlhsy0] - halfOSrhsy0 /.
  Solve[fullOSlhsy0xposv2 == fullOSrhsy0xpos, Q[0, x]][[1]];
halfOSy0 = Collect[%, {Q[0, 0], Q[x, 0], Q0d[1/x], Q1,0, Q2,0, Q3,0}, Factor];
(* let's check it before doing anything else *)
halfOSy0 /. ΔΔ → Δ /. {θ → θs[9], θ0,0 → θs0,0[9]} /. {Q[x, 0] → QQcy[9, 0],
    Q[0, 0] → QQcxy[9, 0, 0], Q0d[1/x] → QQdkeval[9, 0, 1 / x], Q1,0 → QQcxy[9, 1, 0],
    Q2,0 → QQcxy[9, 2, 0], Q3,0 → QQcxy[9, 3, 0]} // Simplificate // Simplify
```

Out[•]= $O[t]^{10}$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
   (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q[x, 0]];
Numerator[%] /. ΔΔ → Δ // FullSimplify // Factor;
Denominator[%%] /. ΔΔ → Δ // FullSimplify // Factor;
%% / % // FullSimplify // Factor;
μx,0 = -%
```

Out[•]= $-2\,c\,(a\,t+x-a\,x)\,\left(-a\,t+a^2\,t+x-a\,x+a^2\,t^2\,x^2\right)$
$\left(-a\,t-b\,t+2\,a\,b\,t+2\,x-a\,x-b\,x+2\,a\,b\,t^2\,x^2\right)\,\left(-b\,t+b^2\,t+x-b\,x+b^2\,t^2\,x^2\right)$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
    (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q0d[1/x]];
ν0d =
 -% /
  Sqrt[
   ΔΔ]
```

Out[•]= $2\,a\,c\,(-a\,t-x+a\,x)\,(-b\,t-x+b\,x)$
$\left(-a\,t+a^2\,t+x-a\,x+a^2\,t^2\,x^2\right)\,\left(-b\,t+b^2\,t+x-b\,x+b^2\,t^2\,x^2\right)$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
    (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q1,0];
Expand[Numerator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ] /
  Expand[Denominator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ];
% /. Sqrt[(1 - t/x)^2 - 4 t^2 x] → Sqrt[ΔΔ] // Factor;
μ1,0 = Factor[% /. ΔΔ → 0]
ν1,0 = Factor[Coefficient[Expand[%%], Sqrt[ΔΔ]]]
```

Out[•]= $-c\,t\,x\,(-a\,t - b\,t + 2\,a\,b\,t + 2\,x - a\,x - b\,x + 2\,a\,b\,t^2\,x^2)\,(2\,a^2\,b\,t^2 - 2\,a^3\,b\,t^2 - 2\,a^2\,b^2\,t^2 + 2\,a^3\,b^2\,t^2 - 2\,a^2\,b\,t\,x + 2\,a^3\,b\,t\,x + 2\,a^2\,b^2\,t\,x - 2\,a^3\,b^2\,t\,x + a^2\,x^2 - a^3\,x^2 - 2\,b\,x^2 + 4\,a\,b\,x^2 - 3\,a^2\,b\,x^2 + a^3\,b\,x^2 + 2\,b^2\,x^2 - 4\,a\,b^2\,x^2 + 2\,a^2\,b^2\,x^2 - 2\,a^3\,b\,t^3\,x^2 - 2\,a^2\,b^2\,t^3\,x^2 + 4\,a^3\,b^2\,t^3\,x^2 - 2\,a^2\,b\,t^2\,x^3 + 2\,a^3\,b\,t^2\,x^3 + 2\,a^2\,b^2\,t^2\,x^3 - 2\,a^3\,b^2\,t^2\,x^3)$

Out[•]= $(a - b)\,c\,t\,x^2\,(2\,a^2\,b\,t^2 - 2\,a^3\,b\,t^2 - 2\,a^2\,b^2\,t^2 + 2\,a^3\,b^2\,t^2 - 2\,a^2\,b\,t\,x + 2\,a^3\,b\,t\,x + 2\,a^2\,b^2\,t\,x - 2\,a^3\,b^2\,t\,x + a^2\,x^2 - a^3\,x^2 - 2\,b\,x^2 + 4\,a\,b\,x^2 - 3\,a^2\,b\,x^2 + a^3\,b\,x^2 + 2\,b^2\,x^2 - 4\,a\,b^2\,x^2 + 2\,a^2\,b^2\,x^2 - 2\,a^3\,b\,t^3\,x^2 - 2\,a^2\,b^2\,t^3\,x^2 + 4\,a^3\,b^2\,t^3\,x^2 - 2\,a^2\,b\,t^2\,x^3 + 2\,a^3\,b\,t^2\,x^3 + 2\,a^2\,b^2\,t^2\,x^3 - 2\,a^3\,b^2\,t^2\,x^3)$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
    (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q2,0];
Expand[Numerator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ] /
  Expand[Denominator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ];
% /. Sqrt[(1 - t/x)^2 - 4 t^2 x] → Sqrt[ΔΔ] // Factor;
μ2,0 = Factor[% /. ΔΔ → 0]
ν2,0 = Factor[Coefficient[Expand[%%], Sqrt[ΔΔ]]]
```

Out[•]= $(-1 + a)\,a^2\,(-1 + b)\,(2 + b)\,c\,t^2\,x^3\,(-a\,t - b\,t + 2\,a\,b\,t + 2\,x - a\,x - b\,x + 2\,a\,b\,t^2\,x^2)$

Out[•]= $-(-1 + a)\,a^2\,(a - b)\,(-1 + b)\,(2 + b)\,c\,t^2\,x^4$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
    (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q3,0];
Expand[Numerator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ] /
  Expand[Denominator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ];
% /. Sqrt[(1 - t/x)^2 - 4 t^2 x] → Sqrt[ΔΔ] // Factor;
μ3,0 = Factor[% /. ΔΔ → 0]
ν3,0 = Factor[Coefficient[Expand[%%], Sqrt[ΔΔ]]]
```

Out[•]= $-2\,(-1 + a)\,a^2\,(-1 + b)\,b\,c\,t^3\,x^3\,(-a\,t - b\,t + 2\,a\,b\,t + 2\,x - a\,x - b\,x + 2\,a\,b\,t^2\,x^2)$

Out[•]= $2\,(-1 + a)\,a^2\,(a - b)\,(-1 + b)\,b\,c\,t^3\,x^4$

In[•]:=
```
Coefficient[halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2)
    (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a x * 2 c, Q[0, 0]];
Expand[Numerator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ] /
   Expand[Denominator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ];
% /. Sqrt[(1 - t/x)^2 - 4 t^2 x] → Sqrt[ΔΔ] // Factor;
μ0,0 = Collect[% /. ΔΔ → 0, θ0,0, Factor]
ν0,0 = Collect[Coefficient[Expand[%%], Sqrt[ΔΔ]], θ0,0, Factor]
```

Out[•]=

$$
\begin{aligned}
&-4 a^3 b^2 t^4 + 4 a^4 b^2 t^4 + 4 a^3 b^3 t^4 - 4 a^4 b^3 t^4 + 4 a^3 b c t^4 - 4 a^4 b c t^4 + 4 a^2 b^2 c t^4 - \\
&12 a^3 b^2 c t^4 + 8 a^4 b^2 c t^4 - 4 a^2 b^3 c t^4 + 8 a^3 b^3 c t^4 - 4 a^4 b^3 c t^4 + 2 a^3 b t^3 x - 2 a^4 b t^3 x + \\
&2 a^2 b^2 t^3 x + 2 a^3 b^2 t^3 x - 4 a^4 b^2 t^3 x - 2 a^2 b^3 t^3 x - 4 a^3 b^3 t^3 x + 6 a^4 b^3 t^3 x - 4 a^3 c t^3 x + \\
&4 a^4 c t^3 x - 6 a^2 b c t^3 x + 11 a^3 b c t^3 x - 5 a^4 b c t^3 x - 2 a b^2 c t^3 x + 5 a^2 b^2 c t^3 x - \\
&3 a^4 b^2 c t^3 x + 2 a b^3 c t^3 x + a^2 b^3 c t^3 x - 7 a^3 b^3 c t^3 x + 4 a^4 b^3 c t^3 x + 2 a^2 b t^2 x^2 - \\
&5 a^3 b t^2 x^2 + 3 a^4 b t^2 x^2 + 2 a b^2 t^2 x^2 - 11 a^2 b^2 t^2 x^2 + 10 a^3 b^2 t^2 x^2 - a^4 b^2 t^2 x^2 - \\
&2 a b^3 t^2 x^2 + 9 a^2 b^3 t^2 x^2 - 5 a^3 b^3 t^2 x^2 - 2 a^4 b^3 t^2 x^2 + 2 a^2 c t^2 x^2 + a^3 c t^2 x^2 - \\
&3 a^4 c t^2 x^2 - 4 a b c t^2 x^2 + 12 a^2 b c t^2 x^2 - 16 a^3 b c t^2 x^2 + 8 a^4 b c t^2 x^2 - 2 b^2 c t^2 x^2 + \\
&15 a b^2 c t^2 x^2 - 28 a^2 b^2 c t^2 x^2 + 20 a^3 b^2 c t^2 x^2 - 5 a^4 b^2 c t^2 x^2 + 2 b^3 c t^2 x^2 - \\
&11 a b^3 c t^2 x^2 + 14 a^2 b^3 c t^2 x^2 - 5 a^3 b^3 c t^2 x^2 + 4 a^4 b^2 t^5 x^2 + 4 a^3 b^3 t^5 x^2 - 8 a^4 b^3 t^5 x^2 - \\
&4 a^4 b c t^5 x^2 - 8 a^3 b^2 c t^5 x^2 + 12 a^4 b^2 c t^5 x^2 - 4 a^2 b^3 c t^5 x^2 + 12 a^3 b^3 c t^5 x^2 - \\
&8 a^4 b^3 c t^5 x^2 - 4 a b t x^3 + 5 a^2 b t x^3 - a^4 b t x^3 + 5 a b^2 t x^3 - 3 a^2 b^2 t x^3 - 3 a^3 b^2 t x^3 + \\
&a^4 b^2 t x^3 - a b^3 t x^3 - 2 a^2 b^3 t x^3 + 3 a^3 b^3 t x^3 + 6 a c t x^3 - 13 a^2 c t x^3 + 8 a^3 c t x^3 - \\
&a^4 c t x^3 + 6 b c t x^3 - 25 a b c t x^3 + 32 a^2 b c t x^3 - 14 a^3 b c t x^3 + a^4 b c t x^3 - 8 b^2 c t x^3 + \\
&22 a b^2 c t x^3 - 20 a^2 b^2 c t x^3 + 6 a^3 b^2 c t x^3 + 2 b^3 c t x^3 - 3 a b^3 c t x^3 + a^2 b^3 c t x^3 - \\
&2 a^4 b t^4 x^3 + 6 a^3 b^2 t^4 x^3 - 8 a^4 b^2 t^4 x^3 - 2 a^2 b^3 t^4 x^3 - 8 a^3 b^3 t^4 x^3 + 14 a^4 b^3 t^4 x^3 + \\
&4 a^4 c t^4 x^3 + 3 a^3 b c t^4 x^3 - 6 a^4 b c t^4 x^3 - 3 a^2 b^2 c t^4 x^3 + 10 a^3 b^2 c t^4 x^3 - 7 a^4 b^2 c t^4 x^3 + \\
&2 a b^3 c t^4 x^3 + 6 a^2 b^3 c t^4 x^3 - 19 a^3 b^3 c t^4 x^3 + 10 a^4 b^3 c t^4 x^3 + 2 a b x^4 - 3 a^2 b x^4 + \\
&a^3 b x^4 - 3 a b^2 x^4 + 4 a^2 b^2 x^4 - a^3 b^2 x^4 + a b^3 x^4 - a^2 b^3 x^4 - 4 c x^4 + 8 a c x^4 - 5 a^2 c x^4 + \\
&a^3 c x^4 + 6 b c x^4 - 11 a b c x^4 + 6 a^2 b c x^4 - a^3 b c x^4 - 2 b^2 c x^4 + 3 a b^2 c x^4 - a^2 b^2 c x^4 - \\
&a^3 b t^3 x^4 - 7 a^2 b^2 t^3 x^4 + 8 a^3 b^2 t^3 x^4 + 3 a^4 b^2 t^3 x^4 - 2 a b^3 t^3 x^4 + 12 a^2 b^3 t^3 x^4 - \\
&9 a^3 b^3 t^3 x^4 - 4 a^4 b^3 t^3 x^4 - a^3 c t^3 x^4 + 8 a^2 b c t^3 x^4 - 11 a^3 b c t^3 x^4 + 4 a^4 b c t^3 x^4 + \\
&9 a b^2 c t^3 x^4 - 29 a^2 b^2 c t^3 x^4 + 24 a^3 b^2 c t^3 x^4 - 5 a^4 b^2 c t^3 x^4 + 2 b^3 c t^3 x^4 - \\
&14 a b^3 c t^3 x^4 + 22 a^2 b^3 c t^3 x^4 - 9 a^3 b^3 c t^3 x^4 - 4 a^4 b^3 t^6 x^4 + 4 a^4 b^2 c t^6 x^4 + \\
&4 a^3 b^3 c t^6 x^4 - 4 a^4 b^3 c t^6 x^4 - 4 a^2 b t^2 x^5 + 7 a^3 b t^2 x^5 - 2 a^4 b t^2 x^5 + 4 a b^2 t^2 x^5 - \\
&a^2 b^2 t^2 x^5 - 6 a^3 b^2 t^2 x^5 + a^4 b^2 t^2 x^5 - 2 a b^3 t^2 x^5 + 3 a^3 b^3 t^2 x^5 - a^3 c t^2 x^5 - 4 a b c t^2 x^5 + \\
&6 a^2 b c t^2 x^5 - 4 b^2 c t^2 x^5 + 13 a b^2 c t^2 x^5 - 12 a^2 b^2 c t^2 x^5 + 2 a^3 b^2 c t^2 x^5 + 2 b^3 c t^2 x^5 - \\
&4 a b^3 c t^2 x^5 + 2 a^2 b^3 c t^2 x^5 - 3 a^4 b^2 t^5 x^5 - 3 a^3 b^3 t^5 x^5 + 8 a^4 b^3 t^5 x^5 - a^4 b c t^5 x^5 + \\
&4 a^3 b^2 c t^5 x^5 - 3 a^4 b^2 c t^5 x^5 + 3 a^2 b^3 c t^5 x^5 - 11 a^3 b^3 c t^5 x^5 + 6 a^4 b^3 c t^5 x^5 - \\
&2 a^4 b t^4 x^6 + 5 a^4 b^2 t^4 x^6 + 2 a^2 b^3 t^4 x^6 - 3 a^3 b^3 t^4 x^6 - 2 a^4 b^3 t^4 x^6 + 2 a^4 c t^4 x^6 - \\
&2 a^3 b c t^4 x^6 - 3 a^4 b c t^4 x^6 - 2 a^2 b^2 c t^4 x^6 + 4 a^3 b^2 c t^4 x^6 + a^4 b^2 c t^4 x^6 - 2 a b^3 c t^4 x^6 + \\
&5 a^2 b^3 c t^4 x^6 - 3 a^3 b^3 c t^4 x^6 + a c t x^3 \left(-a t - b t + 2 a b t + 2 x - a x - b x + 2 a b t^2 x^2\right) \theta_{0,0}
\end{aligned}
$$

Out[ ]= $-a\,x\,\big(2\,a^2\,b^2\,t^3-2\,a^3\,b^2\,t^3-2\,a^2\,b^3\,t^3+2\,a^3\,b^3\,t^3-2\,a^2\,b\,c\,t^3+2\,a^3\,b\,c\,t^3-2\,a\,b^2\,c\,t^3+$
$6\,a^2\,b^2\,c\,t^3-4\,a^3\,b^2\,c\,t^3+2\,a\,b^3\,c\,t^3-4\,a^2\,b^3\,c\,t^3+2\,a^3\,b^3\,c\,t^3-2\,a\,b^2\,t^2\,x+2\,a^3\,b^2\,t^2\,x+$
$2\,a\,b^3\,t^2\,x-2\,a^3\,b^3\,t^2\,x+2\,a^2\,c\,t^2\,x-2\,a^3\,c\,t^2\,x-a^2\,b\,c\,t^2\,x+a^3\,b\,c\,t^2\,x+2\,b^2\,c\,t^2\,x-$
$a\,b^2\,c\,t^2\,x-4\,a^2\,b^2\,c\,t^2\,x+3\,a^3\,b^2\,c\,t^2\,x-2\,b^3\,c\,t^2\,x+a\,b^3\,c\,t^2\,x+3\,a^2\,b^3\,c\,t^2\,x-$
$2\,a^3\,b^3\,c\,t^2\,x-2\,a\,b\,t\,x^2+3\,a^2\,b\,t\,x^2-a^3\,b\,t\,x^2+5\,a\,b^2\,t\,x^2-6\,a^2\,b^2\,t\,x^2+a^3\,b^2\,t\,x^2-$
$3\,a\,b^3\,t\,x^2+3\,a^2\,b^3\,t\,x^2+2\,a\,c\,t\,x^2-5\,a^2\,c\,t\,x^2+3\,a^3\,c\,t\,x^2+2\,b\,c\,t\,x^2-8\,a\,b\,c\,t\,x^2+$
$11\,a^2\,b\,c\,t\,x^2-5\,a^3\,b\,c\,t\,x^2-5\,b^2\,c\,t\,x^2+11\,a\,b^2\,c\,t\,x^2-8\,a^2\,b^2\,c\,t\,x^2+2\,a^3\,b^2\,c\,t\,x^2+$
$3\,b^3\,c\,t\,x^2-5\,a\,b^3\,c\,t\,x^2+2\,a^2\,b^3\,c\,t\,x^2-2\,a^3\,b^2\,t^4\,x^2-2\,a^2\,b^3\,t^4\,x^2+4\,a^3\,b^3\,t^4\,x^2+$
$2\,a^3\,b\,c\,t^4\,x^2+4\,a^2\,b^2\,c\,t^4\,x^2-6\,a^3\,b^2\,c\,t^4\,x^2+2\,a\,b^3\,c\,t^4\,x^2-6\,a^2\,b^3\,c\,t^4\,x^2+$
$4\,a^3\,b^3\,c\,t^4\,x^2+2\,b\,x^3-3\,a\,b\,x^3+a^2\,b\,x^3-3\,b^2\,x^3+4\,a\,b^2\,x^3-a^2\,b^2\,x^3+b^3\,x^3-a\,b^3\,x^3-$
$4\,c\,x^3+7\,a\,c\,x^3-3\,a^2\,c\,x^3+7\,b\,c\,x^3-12\,a\,b\,c\,x^3+5\,a^2\,b\,c\,x^3-3\,b^2\,c\,x^3+5\,a\,b^2\,c\,x^3-$
$2\,a^2\,b^2\,c\,x^3-2\,a^2\,b^2\,t^3\,x^3+4\,a^3\,b^2\,t^3\,x^3+2\,a\,b^3\,t^3\,x^3-4\,a^3\,b^3\,t^3\,x^3-2\,a^3\,c\,t^3\,x^3+$
$3\,a^2\,b\,c\,t^3\,x^3+a\,b^2\,c\,t^3\,x^3-10\,a^2\,b^2\,c\,t^3\,x^3+7\,a^3\,b^2\,c\,t^3\,x^3-2\,b^3\,c\,t^3\,x^3+2\,a\,b^3\,c\,t^3\,x^3+$
$5\,a^2\,b^3\,c\,t^3\,x^3-4\,a^3\,b^3\,c\,t^3\,x^3+a^2\,b\,t^2\,x^4+3\,a\,b^2\,t^2\,x^4-6\,a^2\,b^2\,t^2\,x^4+a^3\,b^2\,t^2\,x^4-$
$2\,a\,b^3\,t^2\,x^4+3\,a^2\,b^3\,t^2\,x^4-3\,a^2\,c\,t^2\,x^4+2\,a^3\,c\,t^2\,x^4-2\,a\,b\,c\,t^2\,x^4+8\,a^2\,b\,c\,t^2\,x^4-$
$4\,a^3\,b\,c\,t^2\,x^4-3\,b^2\,c\,t^2\,x^4+8\,a\,b^2\,c\,t^2\,x^4-8\,a^2\,b^2\,c\,t^2\,x^4+2\,a^3\,b^2\,c\,t^2\,x^4+2\,b^3\,c\,t^2\,x^4-$
$4\,a\,b^3\,c\,t^2\,x^4+2\,a^2\,b^3\,c\,t^2\,x^4+2\,a^3\,b^3\,t^5\,x^4-2\,a^3\,b^2\,c\,t^5\,x^4-2\,a^2\,b^3\,c\,t^5\,x^4+$
$2\,a^3\,b^3\,c\,t^5\,x^4+a^3\,b^2\,t^4\,x^5+a^2\,b^3\,t^4\,x^5-2\,a^3\,b^3\,t^4\,x^5-a^3\,b\,c\,t^4\,x^5-2\,a^2\,b^2\,c\,t^4\,x^5+$
$3\,a^3\,b^2\,c\,t^4\,x^5-a\,b^3\,c\,t^4\,x^5+3\,a^2\,b^3\,c\,t^4\,x^5-2\,a^3\,b^3\,c\,t^4\,x^5\big)-a\,(a-b)\,c\,t\,x^4\,\theta_{0,0}$

```
In[•]:= halfOSy0 * (-a t + a^2 t + x - a x + a^2 t^2 x^2) (-b t + b^2 t + x - b x + b^2 t^2 x^2) * Sqrt[ΔΔ] * a
          x * 2 c /. {Q[x, 0] → 0, Q[0, 0] → 0, Q1,0 → 0, Q2,0 → 0, Q3,0 → 0, Q^d_0[1/x] → 0};
        Expand[Numerator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ] /
          Expand[Denominator[%] * (t - x + x Sqrt[ΔΔ]) /. ΔΔ → Δ];
        % /. Sqrt[(1 - t/x)^2 - 4 t^2 x] → Sqrt[ΔΔ] // Factor;
        μ = Collect[% /. ΔΔ → 0, θ, Factor]
        ν = Collect[Coefficient[Expand[%%], Sqrt[ΔΔ]], θ, Factor]
```

$$\text{Out}=a\,b\,\big(4a^2bt^4-4a^3bt^4-4a^2b^2t^4+4a^3b^2t^4-2a^2t^3x+2a^3t^3x-2abt^3x-2a^2bt^3x+4a^3bt^3x+2ab^2t^3x+4a^2b^2t^3x-6a^3b^2t^3x-2at^2x^2+5a^2t^2x^2-3a^3t^2x^2-2bt^2x^2+11abt^2x^2-10a^2bt^2x^2+a^3bt^2x^2+2b^2t^2x^2-9ab^2t^2x^2+5a^2b^2t^2x^2+2a^3b^2t^2x^2-4a^3bt^5x^2-4a^2b^2t^5x^2+8a^3b^2t^5x^2+4tx^3-5atx^3+a^3tx^3-5btx^3+3abtx^3+3a^2btx^3-a^3btx^3+b^2tx^3+2ab^2tx^3-3a^2b^2tx^3+2a^3t^4x^3-6a^2bt^4x^3+8a^3bt^4x^3+2ab^2t^4x^3+8a^2b^2t^4x^3-14a^3b^2t^4x^3-2x^4+3ax^4-a^2x^4+3bx^4-4abx^4+a^2bx^4-b^2x^4+ab^2x^4+a^2t^3x^4+7abt^3x^4-8a^2bt^3x^4-3a^3bt^3x^4+2b^2t^3x^4-12ab^2t^3x^4+9a^2b^2t^3x^4+4a^3b^2t^3x^4+4a^3b^2t^6x^4+4at^2x^5-7a^2t^2x^5+2a^3t^2x^5-4bt^2x^5+abt^2x^5+6a^2bt^2x^5-a^3bt^2x^5+2b^2t^2x^5-3a^2b^2t^2x^5+3a^3bt^5x^5+3a^2b^2t^5x^5-8a^3b^2t^5x^5+2a^3t^4x^6-5a^3bt^4x^6-2ab^2t^4x^6+3a^2b^2t^4x^6+2a^3b^2t^4x^6\big)+a\,c\,t\,x^3\,(-at-bt+2abt+2x-ax-bx+2abt^2x^2)\,\theta$$

$$\text{Out}=a\,b\,x\,\big(2a^2bt^3-2a^3bt^3-2a^2b^2t^3+2a^3b^2t^3-2abt^2x+2a^3bt^2x+2ab^2t^2x-2a^3b^2t^2x-2atx^2+3a^2tx^2-a^3tx^2+5abtx^2-6a^2btx^2+a^3btx^2-3ab^2tx^2+3a^2b^2tx^2-2a^3bt^4x^2-2a^2b^2t^4x^2+4a^3b^2t^4x^2+2x^3-3ax^3+a^2x^3-3bx^3+4abx^3-a^2bx^3+b^2x^3-ab^2x^3-2a^2bt^3x^3+4a^3bt^3x^3+2ab^2t^3x^3-4a^3b^2t^3x^3+a^2t^2x^4+3abt^2x^4-6a^2bt^2x^4+a^3bt^2x^4-2ab^2t^2x^4+3a^2b^2t^2x^4+2a^3b^2t^5x^4+a^3bt^4x^5+a^2b^2t^4x^5-2a^3b^2t^4x^5\big)-a\,(a-b)\,c\,t\,x^4\,\theta$$

```
In[•]:= (* check it all *)
        -μx,0 Q[x, 0] - ν^d_0 Sqrt[ΔΔ] Q^d_0[1/x] + (μ + ν Sqrt[ΔΔ]) + (μ0,0 + ν0,0 Sqrt[ΔΔ]) Q[0, 0] +
          (μ1,0 + ν1,0 Sqrt[ΔΔ]) Q1,0 + (μ2,0 + ν2,0 Sqrt[ΔΔ]) Q2,0 + (μ3,0 + ν3,0 Sqrt[ΔΔ]) Q3,0;
        % /. ΔΔ → Δ /. {θ → θs[12], θ0,0 → θs0,0[12]} /.
           {Q[x, 0] → QQcy[12, 0], Q^d_0[1/x] → QQdkeval[12, 0, 1/x],
            Q[0, 0] → QQcxy[12, 0, 0], Q1,0 → QQcxy[12, 1, 0], Q2,0 → QQcxy[12, 2, 0],
            Q3,0 → QQcxy[12, 3, 0]} // Simplificate // Simplify
Out[•]= O[t]^13
```

In[∘]:=
```
(* now for the canonical factorisation (5.21)-(5,23)*)
Off[Root::sbr]
d1 = Root[t^2 - 2 t # + #^2 - 4 t^2 #^3 &, 1];
d2 = Root[t^2 - 2 t # + #^2 - 4 t^2 #^3 &, 2];
d3 = Root[t^2 - 2 t # + #^2 - 4 t^2 #^3 &, 3];
X1 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 3}]] == t + 2 t^(5/2) &][[1]]
X2 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 3}]] == t - 2 t^(5/2) &][[1]]
X3 = Select[{d1, d2, d3}, Normal[Series[#, {t, 0, 1}]] == 1/(4 t^2) - 2 t &][[1]]
Series[{X1, X2, X3}, {t, 0, 10}]
```

Out[∘]= $\mathrm{Root}\left[-t^2+2\,t\,\#1-\#1^2+4\,t^2\,\#1^3\,\&,\,2\right]$

Out[∘]= $\mathrm{Root}\left[-t^2+2\,t\,\#1-\#1^2+4\,t^2\,\#1^3\,\&,\,1\right]$

Out[∘]= $\mathrm{Root}\left[-t^2+2\,t\,\#1-\#1^2+4\,t^2\,\#1^3\,\&,\,3\right]$

Out[∘]=
$$\left\{t+2\,t^{5/2}+6\,t^4+21\,t^{11/2}+80\,t^7+\frac{1287\,t^{17/2}}{4}+1344\,t^{10}+O[t]^{21/2},\right.$$
$$t-2\,t^{5/2}+6\,t^4-21\,t^{11/2}+80\,t^7-\frac{1287\,t^{17/2}}{4}+1344\,t^{10}+O[t]^{21/2},$$
$$\left.\frac{1}{4\,t^2}-2\,t-12\,t^4-160\,t^7-2688\,t^{10}+O[t]^{11}\right\}$$

In[∘]:=
```
(* then the factorisation (5.24)-(5.26)*)
Δm = (1 - X1 / x) (1 - X2 / x);
Δp = 1 - x / X3;
Δ0 = 4 t^2 X3;
(* so that *)
Δ - Δ0 Δp Δm // FullSimplify
```

Out[∘]= 0

In[∘]:=
```
(* so that (5.27)-(5.28)*)
Series[1 / Sqrt[Δp], {t, 0, 5}]
Series[Sqrt[Δ0 Δm], {t, 0, 5}]
```

Out[∘]= $1+2\,x\,t^2+6\,x^2\,t^4+16\,x\,t^5+O[t]^6$

Out[∘]= $1-\frac{t}{x}-4\,t^3-\frac{2\,t^4}{x}-\frac{2\,t^5}{x^2}+O[t]^6$

In[∘]:=
```
(* now we divide by Sqrt[Δp] and take the [x^>] and [x^<] parts *)
(* for simplicity define *)
ΔΔm = (1 - XX1 / x) (1 - XX2 / x);
ΔΔp = 1 - x / XX3;
ΔΔ0 = 4 t^2 XX3;
```

In[•]:=
```
(* the following two expansions will be useful *)
(* the expansion of 1/Sqrt[Δ₊] *)
Series[1 / Sqrt[ΔΔp], {x, 0, 5}]
(* and the expansion of Sqrt[Δ₋] *)
Series[Sqrt[ΔΔm], {x, Infinity, 5}];
ApplyToSeries[Factor, %]
```

Out[•]=
$$1+\frac{x}{2\,XX_3}+\frac{3\,x^2}{8\,XX_3^2}+\frac{5\,x^3}{16\,XX_3^3}+\frac{35\,x^4}{128\,XX_3^4}+\frac{63\,x^5}{256\,XX_3^5}+O[x]^6$$

Out[•]=
$$1+\frac{-XX_1-XX_2}{2\,x}-\frac{(XX_1-XX_2)^2}{8\,x^2}-\frac{(XX_1-XX_2)^2\,(XX_1+XX_2)}{16\,x^3}-\frac{(XX_1-XX_2)^2\,(5\,XX_1^2+6\,XX_1\,XX_2+5\,XX_2^2)}{128\,x^4}-\frac{(XX_1-XX_2)^2\,(XX_1+XX_2)\,(7\,XX_1^2+2\,XX_1\,XX_2+7\,XX_2^2)}{256\,x^5}+O\left[\frac{1}{x}\right]^6$$

In[•]:=
```
(* first take the [x^>] part *)
(* unfortunately no matter what we do we will end up with
 another unknown -- the simplest route involves dividing by x^4 *)
```

In[•]:=
```
(* Q[x,0] term is straightforward *)
μx,0 / x^4 / Sqrt[ΔΔp];
xposLHS1 = % * Q[x, 0] - SeriesCoefficient[%, {x, 0, -4}] / x^4 *
      (Q[0, 0] + Q1,0 * x + Q2,0 * x^2 + Q3,0 * x^3 + Q4,0 * x^4) -
    SeriesCoefficient[%, {x, 0, -3}] / x^3 * (Q[0, 0] + Q1,0 * x + Q2,0 * x^2 + Q3,0 * x^3) -
    SeriesCoefficient[%, {x, 0, -2}] / x^2 * (Q[0, 0] + Q1,0 * x + Q2,0 * x^2) -
    SeriesCoefficient[%, {x, 0, -1}] / x^1 * (Q[0, 0] + Q1,0 * x) -
    SeriesCoefficient[%, {x, 0, 0}] * Q[0, 0] // Simplify
(* check it *)
μx,0 / x^4 / Sqrt[Δp] * Q[x, 0] /. {Q[x, 0] → QQcy[9, 0]};
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposLHS1 /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
   {Q[x, 0] → QQcy[9, 0], Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0],
    Q2,0 → QQcxy[9, 2, 0], Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0]};
% - %% // Simplify
```

Out[•]=
$$\frac{1}{64\,XX_3^4}\,c\,\Big(-16\,XX_3^2\,\big(-16\,XX_3^2+12\,b\,XX_3\,(t+2\,XX_3)-$$
$$b^2\,(3\,t^2+16\,t\,XX_3+8\,XX_3^2)+b^3\,t\,(3\,t+4\,XX_3+8\,t^2\,XX_3^2)\big)+$$
$$16\,a\,XX_3^2\,\big(-4\,XX_3\,(t+10\,XX_3)+b\,(3\,t^2+40\,t\,XX_3+56\,XX_3^2)+$$
$$4\,b^3\,t\,(3\,t+2\,XX_3+12\,t^2\,XX_3^2)-b^2\,(15\,t^2+44\,t\,XX_3+16\,XX_3^2+24\,t^3\,XX_3^2)\big)+$$
$$a^4\,t\,\big(-8\,(5\,t^2\,XX_3-8\,XX_3^3+8\,t^3\,XX_3^3)-3\,b^2\,t\,(35\,t^2-48\,XX_3^2+96\,t^3\,XX_3^2-256\,t\,XX_3^4+$$
$$128\,t^4\,XX_3^4)+2\,b^3\,t^2\,(-40\,XX_3+144\,t^2\,XX_3^2+384\,t^3\,XX_3^4+t\,(35-192\,XX_3^3))+$$
$$b\,(-144\,t\,XX_3^2+48\,t^4\,XX_3^2-64\,XX_3^3-40\,t^2\,XX_3\,(-3+16\,XX_3^3)+5\,t^3\,(7+64\,XX_3^3))\big)+$$
$$a^2\,\big(32\,XX_3^2\,(-3\,t^2+8\,t\,XX_3+16\,XX_3^2)+b^3\,t\,(-288\,t\,XX_3^2+48\,t^4\,XX_3^2-64\,XX_3^3-1408\,t^2\,XX_3^4+$$
$$t^3\,(35-64\,XX_3^3))-8\,b\,XX_3\,(12\,t^2\,XX_3+112\,t\,XX_3^2+80\,XX_3^3+3\,t^3\,(-5+16\,XX_3^3))+$$
$$b^2\,(480\,t^2\,XX_3^2+704\,t\,XX_3^3+128\,XX_3^4+24\,t^3\,XX_3\,(-5+64\,XX_3^3)-t^4\,(35+64\,XX_3^3))\big)+$$
$$a^3\,\big(8\,XX_3\,(12\,t^2\,XX_3-32\,t\,XX_3^2-16\,XX_3^3+t^3\,(5+16\,XX_3^3))+b^3\,t^2$$

$$\left(144\,XX_3^2 - 288\,t^3\,XX_3^2 - 384\,t^4\,XX_3^4 + 16\,t\,XX_3\,\left(5 + 48\,XX_3^3\right) + 3\,t^2\,\left(-35 + 128\,XX_3^3\right)\right) +$$
$$b\,\left(192\,t^2\,XX_3^2 + 512\,t\,XX_3^3 + 128\,XX_3^4 + 48\,t^3\,XX_3\,\left(-5 + 16\,XX_3^3\right) - t^4\,\left(35 + 256\,XX_3^3\right)\right) +$$
$$4\,b^2\,t\,\left(-108\,t\,XX_3^2 + 48\,t^4\,XX_3^2 - 64\,XX_3^3 + t^3\,\left(35 + 32\,XX_3^3\right) + t^2\,\left(30\,XX_3 - 448\,XX_3^4\right)\right)\Big)\Big)$$
$$Q[0,\,0] + \frac{1}{8\,x\,XX_3^3}\,c\,t\,\Big(-8\,\left(-1 + b\right)\,b\,XX_3^2\,\left(-6\,XX_3 + b\,\left(t + 2\,XX_3\right)\right) +$$
$$8\,a\,\left(-1 + b\right)\,XX_3^2\,\left(2\,XX_3 + 4\,b^2\,\left(t + XX_3\right) - b\,\left(t + 18\,XX_3\right)\right) +$$
$$a^2\,\Big(16\,XX_3^2\,\left(-t + 4\,XX_3\right) - 2\,b\,XX_3\,\left(-9\,t^2 + 8\,t\,XX_3 + 112\,XX_3^2\right) +$$
$$b^3\,\left(-48\,t\,XX_3^2 + 8\,t^4\,XX_3^2 - 16\,XX_3^3 + t^3\,\left(5 - 16\,XX_3^3\right)\right) +$$
$$b^2\,\left(-18\,t^2\,XX_3 + 80\,t\,XX_3^2 + 176\,XX_3^3 - t^3\,\left(5 + 16\,XX_3^3\right)\right)\Big) +$$
$$a^3\,\Big(2\,XX_3\,\left(3\,t^2 + 8\,t\,XX_3 - 32\,XX_3^2\right) - 3\,b^3\,t\,\left(-4\,t\,XX_3 - 8\,XX_3^2 + 16\,t^3\,XX_3^2 + t^2\,\left(5 - 32\,XX_3^3\right)\right) +$$
$$2\,b^2\,\left(9\,t^2\,XX_3 - 36\,t\,XX_3^2 + 16\,t^4\,XX_3^2 - 32\,XX_3^3 + 2\,t^3\,\left(5 + 8\,XX_3^3\right)\right) +$$
$$b\,\left(-36\,t^2\,XX_3 + 32\,t\,XX_3^2 + 128\,XX_3^3 - t^3\,\left(5 + 64\,XX_3^3\right)\right)\Big) +$$
$$a^4\,\Big(-2\,XX_3\,\left(3\,t^2 - 8\,XX_3^2 + 8\,t^3\,XX_3^2\right) - 3\,b^2\,\left(5\,t^3 - 8\,t\,XX_3^2 + 16\,t^4\,XX_3^2\right) +$$
$$2\,b^3\,t^2\,\left(-6\,XX_3 + 24\,t^2\,XX_3^2 + t\,\left(5 - 48\,XX_3^3\right)\right) +$$
$$b\,\left(18\,t^2\,XX_3 - 24\,t\,XX_3^2 + 8\,t^4\,XX_3^2 - 16\,XX_3^3 + t^3\,\left(5 + 80\,XX_3^3\right)\right)\Big)\Big)\,\left(Q_{1,0}\,x + Q[0,\,0]\right) +$$
$$\frac{1}{4\,x^2\,XX_3^2}\,c\,t^2\,\Big(-8\,\left(-1 + b\right)\,b^2\,XX_3^2 + 8\,a\,b\,\left(1 - 5\,b + 4\,b^2\right)\,XX_3^2 +$$
$$a^2\,\Big(4\,b\,\left(3\,t - 4\,XX_3\right)\,XX_3 - 16\,XX_3^2 + b^2\,\left(-3\,t^2 - 12\,t\,XX_3 + 80\,XX_3^2\right) +$$
$$b^3\,\left(3\,t^2 - 48\,XX_3^2 + 8\,t^3\,XX_3^2\right)\Big) + a^3\,\Big(4\,XX_3\,\left(t + 4\,XX_3\right) + b\,\left(-3\,t^2 - 24\,t\,XX_3 + 32\,XX_3^2\right) +$$
$$b^3\,\left(-9\,t^2 + 8\,t\,XX_3 + 24\,XX_3^2 - 48\,t^3\,XX_3^2\right) + 4\,b^2\,\left(3\,t^2 + 3\,t\,XX_3 - 18\,XX_3^2 + 8\,t^3\,XX_3^2\right)\Big) +$$
$$a^4\,\Big(-4\,t\,XX_3 + 2\,b^3\,t\,\left(3\,t - 4\,XX_3 + 24\,t^2\,XX_3^2\right) + b\,\left(3\,t^2 + 12\,t\,XX_3 - 24\,XX_3^2 + 8\,t^3\,XX_3^2\right) -$$
$$3\,b^2\,\left(3\,t^2 - 8\,XX_3^2 + 16\,t^3\,XX_3^2\right)\Big)\Big)$$
$$\left(Q_{1,0}\,x + Q_{2,0}\,x^2 + Q[0,\,0]\right) + \frac{1}{x^3\,XX_3}\,(-1 + a)\,a^2\,\left(-1 + b\right)\,c\,t^3$$
$$\left(b\,\left(-b\,t + 6\,XX_3\right) + a\,\left(2\,b^2\,\left(t - 2\,XX_3\right) + 2\,XX_3 - b\,\left(t + 4\,XX_3\right)\right)\right)$$
$$\big(Q_{1,0}\,x + Q_{2,0}\,x^2 +$$
$$Q_{3,0}\,x^3 + Q[0,\,0]\big) +$$
$$\frac{2\,(-1 + a)\,a^2\,\left(-1 + b\right)\,b\,\left(-b + a\,\left(-1 + 2\,b\right)\right)\,c\,t^4\,\left(Q_{1,0}\,x + Q_{2,0}\,x^2 + Q_{3,0}\,x^3 + Q_{4,0}\,x^4 + Q[0,\,0]\right)}{x^4} -$$
$$\frac{1}{x^4\,\sqrt{1 - \frac{x}{XX_3}}}$$
$$2$$
$$c$$
$$(a\,(t - x) + x)$$
$$\left(x - a\,(t + x) + a^2\,t\,\left(1 + t\,x^2\right)\right)$$
$$\left(x - b\,(t + x) + b^2\,t\,\left(1 + t\,x^2\right)\right)$$
$$\left(2\,x - b\,(t + x) + a\,\left(\left(-1 + 2\,b\right)\,t - x + 2\,b\,t^2\,x^2\right)\right)$$
$$Q[$$
$$x,\,0]$$

Out[∘]= $O[t]^{10}$

In[•]:=
```
(* then the Q_0^d[1/x] term *)
ν_0^d / x^4 * Sqrt[ΔΔ_0 ΔΔ_m];
SeriesCoefficient[%, {x, Infinity, -2}] * x^2 * (Q[0, 0] + Q_{1,1} / x) +
  SeriesCoefficient[%, {x, Infinity, -1}] * x * (Q[0, 0]);
% /. Solve[Q10eqn == 0, Q_{1,1}][[1]];
xposLHS2 = % /. Sqrt[t^2 XX_3] → t Sqrt[XX_3] // Simplify
(* check it *)
ν_0^d / x^4 * Sqrt[Δ_0 Δ_m] * Q_0^d[1 / x] /. {Q_0^d[1 / x] → QQdkeval[9, 0, 1 / x]};
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
xposLHS2 /. {XX_1 → X_1, XX_2 → X_2, XX_3 → X_3} /.
  {Q[0, 0] → QQcxy[9, 0, 0], Q_{1,0} → QQcxy[9, 1, 0], Q_{2,0} → QQcxy[9, 2, 0]};
% - %% // Simplify[#, Assumptions → t > 0] &
```

Out[•]=

$$2\,a\,c\,t^3\,x\,\sqrt{XX_3}\,\Big(\big(\big(-2\,(-1+a)\,a^2\,(-1+b)^2+2\,b^2\,\big(-(-1+a)^2\,(-1+b)-a^2\,(-b+a\,(-1+2\,b))\big)\,t^3\big)-(-1+a)\,a^2\,(-1+b)\,b^2\,t^2\,(XX_1+XX_2)\big)\,Q[0,0]+2\,(-1+a)\,a\,(-1+b)\,b^2\,t\,\big(Q_{1,0}+a\,t\,(-Q_{2,0}+x\,Q[0,0])\big)\Big)$$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* then the Q_{1,0} term *)
μ_{1,0} / x^4 / Sqrt[ΔΔ_p];
% - (Series[%, {x, 0, 0}] // Normal) // Simplify;
ν_{1,0} / x^4 * Sqrt[ΔΔ_0 ΔΔ_m];
(* the addition of the (O[x,Infinity]^1)*x term is because some versions of
 Mathematica seem to include unwanted terms when expanding around infinity *)
Series[%, {x, Infinity, -1}] + (O[x, Infinity]^1) * x /.
   Sqrt[t^2 XX_3] → t Sqrt[XX_3] // Normal;
xposRHS1 = (%%% + % // Simplify) * Q_{1,0}
(* check it *)
Series[(μ_{1,0} + ν_{1,0} Sqrt[Δ]) / x^4 / Sqrt[Δ_p], {t, 0, 10}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
Series[xposRHS1 / Q_{1,0} /. {XX_1 → X_1, XX_2 → X_2, XX_3 → X_3}, {t, 0, 10}];
% - %% // Simplify[#, Assumptions → t > 0] &
```

Out[•]=

$$
\begin{aligned}
&c\, Q_{1,0}\, t \\
&\Bigg(-4\,(-1+a)\,a^3\,(-1+b)\,b^2\,t^3-2\,(-1+a)\,a^2\,(-1+b)\,b\,(-b+a\,(-1+2\,b))\,t^3-(-2+a+b) \\
&\quad (2\,(-1+b)\,b-4\,a\,(-1+b)\,b+a^3\,(-1+b-2\,b\,t^3+4\,b^2\,t^3)+a^2\,(1-3\,b-2\,b^2\,(-1+t^3)))+ \\
&\quad \frac{2\,(-1+a)\,a^2\,(-1+b)\,b\,(-b+a\,(-1+2\,b))\,t^3}{x^3}+\frac{1}{x^3\,\sqrt{1-\frac{x}{XX_3}}} \\
&\quad (2\,x-b\,(t+x)+a\,((-1+2\,b)\,t-x+2\,b\,t^2\,x^2))\,(-2\,(-1+b)\,b\,x^2+4\,a\,(-1+b)\,b\,x^2+ \\
&\qquad a^3\,(x^2-2\,b^2\,t\,(t-x+2\,t^2\,x^2-t\,x^3)+b\,(-2\,t\,x-x^2+2\,t^3\,x^2-2\,t^2\,(-1+x^3)))+ \\
&\qquad a^2\,(-x^2+2\,b^2\,(-t\,x-x^2+t^3\,x^2-t^2\,(-1+x^3))+b\,(2\,t\,x+3\,x^2+2\,t^2\,(-1+x^3))))+ \\
&\quad \frac{5\,(-1+a)\,a^2\,(-1+b)\,b\,(-b+a\,(-1+2\,b))\,t^3}{8\,XX_3^3}-\frac{3\,(-1+a)\,a^2\,(-1+b)\,b\,(-1+a\,b)\,t^2}{2\,XX_3^2}+ \\
&\quad \frac{1}{2\,XX_3}\,t\,(-2\,(-1+b)\,b^2+2\,a\,b\,(1-5\,b+4\,b^2)+a^2\,b\,(-9+21\,b+2\,b^2\,(-6+t^3))+ \\
&\qquad a^4\,(1+12\,b^3\,t^3+b^2\,(4-12\,t^3)+b\,(-5+2\,t^3))+ \\
&\qquad a^3\,(-1+12\,b+b^3\,(6-12\,t^3)+b^2\,(-17+8\,t^3)))-4\,(-1+a)\,a^2\,(a-b)\,(-1+b)\,b \\
&\quad t^3\,x\,\sqrt{XX_3}+\frac{(-1+a)\,a^2\,(-1+b)\,b\,t^2\,(-b\,t+a\,(-1+2\,b)\,t+4\,XX_3-4\,a\,b\,XX_3)}{x^2\,XX_3}+ \\
&\quad \frac{1}{4\,x\,XX_3^2}\,t\,(-8\,(-1+b)\,b^2\,XX_3^2+8\,a\,b\,(1-5\,b+4\,b^2)\,XX_3^2+ \\
&\qquad a^2\,b\,(4\,(2\,t-9\,XX_3)\,XX_3+b\,(-3\,t^2-8\,t\,XX_3+84\,XX_3^2)+b^2\,(3\,t^2-48\,XX_3^2+8\,t^3\,XX_3^2))+ \\
&\qquad a^4\,(4\,XX_3^2+2\,b^3\,t\,(3\,t-4\,XX_3+24\,t^2\,XX_3^2)+b^2\,(-9\,t^2+8\,t\,XX_3+16\,XX_3^2-48\,t^3\,XX_3^2)+ \\
&\qquad\quad b\,(3\,t^2-20\,XX_3^2+8\,t^3\,XX_3^2))+a^3\,(-4\,XX_3^2+b\,(-3\,t^2-8\,t\,XX_3+48\,XX_3^2)+ \\
&\qquad\quad b^3\,(-9\,t^2+8\,t\,XX_3+24\,XX_3^2-48\,t^3\,XX_3^2)+4\,b^2\,(3\,t^2-17\,XX_3^2+8\,t^3\,XX_3^2)))\Bigg)
\end{aligned}
$$

Out[•]= $O[t]^{11}$

In[•]:=
```
(* then the Q2,0 term *)
μ2,0 / x^4 / Sqrt[ΔΔp];
% - (Series[%, {x, 0, 0}] // Normal) // Simplify;
ν2,0 / x^4 * Sqrt[ΔΔ0 ΔΔm];
(* the addition of the (O[x,Infinity]^1)*x term is because some versions of
 Mathematica seem to include unwanted terms when expanding around infinity *)
(Series[%, {x, Infinity, -1}] + (O[x, Infinity]^1) * x) /.
    Sqrt[t^2 XX3] → t Sqrt[XX3] // Normal;
xposRHS2 = (%%% + % // Simplify) * Q2,0
(* check it *)
Series[(μ2,0 + ν2,0 Sqrt[Δ]) / x^4 / Sqrt[Δp], {t, 0, 10}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
Series[xposRHS2 / Q2,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 10}];
% - %% // Simplify[#, Assumptions → t > 0] &
```

Out[•]=
$$(-1+a)\, a^2\, (-1+b)\, (2+b)\, c\, Q_{2,0}\, t^2 \left(-2+a+b+\frac{(a+b-2ab)\, t}{x} + \frac{2x - b\,(t+x) + a\left((-1+2b)\, t - x + 2b\, t^2 x^2\right)}{x\sqrt{1-\frac{x}{XX_3}}} + \frac{(a+b-2ab)\, t}{2\, XX_3} \right)$$

Out[•]= $O[t]^{11}$

In[•]:=
```
(* then the Q3,0 term *)
μ3,0 / x^4 / Sqrt[ΔΔp];
% - (Series[%, {x, 0, 0}] // Normal) // Simplify;
ν3,0 / x^4 * Sqrt[ΔΔ0 ΔΔm];
(* the addition of the (O[x,Infinity]^1)*x term is because some versions of
 Mathematica seem to include unwanted terms when expanding around infinity *)
(Series[%, {x, Infinity, -1}] + (O[x, Infinity]^1) * x) /.
    Sqrt[t^2 XX3] → t Sqrt[XX3] // Normal;
xposRHS3 = (%%% + % // Simplify) * Q3,0
(* check it *)
Series[(μ3,0 + ν3,0 Sqrt[Δ]) / x^4 / Sqrt[Δp], {t, 0, 10}];
ApplyToSeries[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &] &, %];
Series[xposRHS3 / Q3,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 10}];
% - %% // Simplify[#, Assumptions → t > 0] &
```

Out[•]=

$$(-1+a)\,a^2\,(-1+b)\,b\,c\,Q_{3,0}\,t^3\left(\frac{2\,(-b+a\,(-1+2\,b))\,t}{x}+\frac{2\,\left(-2\,x+b\,(t+x)+a\,\left(t-2\,b\,t+x-2\,b\,t^2\,x^2\right)\right)}{x\,\sqrt{1-\frac{x}{XX_3}}}-\frac{-4\,XX_3+b\,(t+2\,XX_3)+a\,(t-2\,b\,t+2\,XX_3)}{XX_3}\right)$$

Out[•]= $O[t]^{11}$

In[•]:=
```
(* now the constant term and the Q[0,0] coefficient are not so nice,
since they contain non-algebraic terms *)
(* in particular ν and ν0,0 cause trouble because they lead to
 (series with 1/x coefficients)*(series with x coefficients) *)
(* so the best we can do is give them a name or evaluate them manually *)
```

In[•]:=
```
(* the constant term*)
xposRHS4s[N_] := ApplyToSeries[
   Factor[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &]] &,
   μ / x^4 / Sqrt[Δp] + ν / x^4 * Sqrt[Δ0 Δm] /. θ → θs[N]]
```

In[•]:=
```
(* the Q[0,0] term*)
xposRHS5s[N_] := ApplyToSeries[
    Factor[Select[Expand[#] + x^(-π) + x^(-2 π), Exponent[#, x] > 0 &]] &,
    μ0,0 / x^4 / Sqrt[Δp] + ν0,0 / x^4 * Sqrt[Δ0 Δm] /. θ0,0 → θs0,0[N]] * Q[0, 0]
```

In[•]:=
```
xposRHS4s[3]
xposRHS5s[3]
```

Out[•]=
$$2\ (-1+a)^2\ a\ b\ (-2+a+b)\ x\ t^2 - 2\ \left((-1+a)\ a\ b\ \left(-6+7\ a-2\ a^2+5\ b-7\ a\ b+3\ a^2\ b-b^2+a\ b^2\right)\ x^2\right)\ t^4+O[t]^5$$

Out[•]=
$$-2\ \left((-1+a)^2\ \left(-2\ a\ b+a^2\ b+a\ b^2+4\ c+a^2\ c-6\ b\ c+ a\ b\ c-2\ a^2\ b\ c+4\ b^2\ c-3\ a\ b^2\ c+a^2\ b^2\ c-b^3\ c+a\ b^3\ c\right)\ x\ Q[0,\ 0]\right)\ t^2 + 2\ (-1+a)\ \left(-6\ a\ b+7\ a^2\ b-2\ a^3\ b+5\ a\ b^2-7\ a^2\ b^2+3\ a^3\ b^2-a\ b^3+a^2\ b^3+12\ c- 12\ a\ c+3\ a^2\ c-18\ b\ c+19\ a\ b\ c-5\ a^2\ b\ c+10\ b^2\ c-11\ a\ b^2\ c+3\ a^2\ b^2\ c- a^3\ b^2\ c-2\ b^3\ c+3\ a\ b^3\ c-2\ a^2\ b^3\ c+a^3\ b^3\ c\right)\ x^2\ Q[0,\ 0]\ t^4+O[t]^5$$

In[•]:=
```
(* then constructing equation (5.29) *)
(* multiplying everything by x^4 keeps the
  powers of x in the Q[0,0] coefficient non-negative *)
P_{x,0} = -(-a t + a^2 t + x - a x + a^2 t^2 x^2) (-a t - b t + 2 a b t + 2 x - a x - b x + 2 a b t^2 x^2)
   (-b t + b^2 t + x - b x + b^2 t^2 x^2);
σ_{x,0} = x^4 * Factor[Coefficient[xposLHS1, Q[x, 0]] / P_{x,0}];
σ_{1,0} =
  x^4 * Coefficient[-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3, Q_{1,0}] //
   Factor;
σ_{2,0} = x^4 * Coefficient[-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3, Q_{2,0}] //
   Factor;
σ_{3,0} = x^4 * Coefficient[-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3, Q_{3,0}] //
   Factor;
σ_{4,0} = x^4 * Coefficient[-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3, Q_{4,0}] //
   Factor;
```

In[•]:=
```
(* these are unwieldy and it will be more useful to have series expansions *)
(* first define these *)
Clear[Xs1, Xs2, Xs3, σsx,0, σs1,0, σs2,0, σs3,0, σs, σs0,0, xpos00]
Xs1[n_] := Xs1[n] = Series[X1, {t, 0, n}]
Xs2[n_] := Xs2[n] = Series[X2, {t, 0, n}]
Xs3[n_] := Xs3[n] = Series[X3, {t, 0, n}]
(* then *)
σsx,0[n_] := σsx,0[n] = ApplyToSeries[Factor,
   Simplify[σx,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
σs1,0[n_] := σs1,0[n] = ApplyToSeries[Factor,
   Simplify[σ1,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
σs2,0[n_] := σs2,0[n] = ApplyToSeries[Factor,
   Simplify[σ2,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
σs3,0[n_] := σs3,0[n] = ApplyToSeries[Factor,
   Simplify[σ3,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
(* σ4,0 is a polynomial so we don't need to do anything with it *)
(* these will also be useful *)
xpos00[n_] := xpos00[n] = ApplyToSeries[Factor,
   Coefficient[-xposLHS1 - xposLHS2 + xposRHS1 + xposRHS2 + xposRHS3, Q[0, 0]] /.
     {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} /. Sqrt[1/t^2] → 1 / t]
(* then *)
σs[n_] := σs[n] = x^4 * ApplyToSeries[Factor,
    xposRHS4s[n] /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} // Simplificate]
σs0,0[n_] := σs0,0[n] = x^4 * ApplyToSeries[Factor,
    (xposRHS5s[n] / Q[0, 0] + xpos00[n]) /.
      {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} // Simplificate]
```

In[•]:=
```
(* check it *)
-σsx,0[9] Px,0 Q[x, 0] + σs1,0[9] Q1,0 +
   σs2,0[9] Q2,0 + σs3,0[9] Q3,0 + σ4,0 Q4,0 + σs[9] + σs0,0[9] Q[0, 0] /.
  {Q[x, 0] → QQcy[9, 0], Q1,0 → QQcxy[9, 1, 0], Q2,0 → QQcxy[9, 2, 0],
   Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0], Q[0, 0] → QQcxy[9, 0, 0]};
% // Simplificate;
ApplyToSeries[Factor, %]
```

Out[•]= $O[t]^{10}$

In[•]:=
```
(* next for the [x^<] part *)
(* things will be simplest if we divide by x^5 first *)
(* start with the Q0d part *)
ν0d / x^5 * Sqrt[ΔΔ0 ΔΔm];
% * Q0d[1 / x] - SeriesCoefficient[%, {x, Infinity, -1}] * x * (Q[0, 0] + Q1,1 / x) -
  SeriesCoefficient[%, {x, Infinity, 0}] * Q[0, 0];
Simplify[%, Assumptions → t > 0 && x > 0];
xnegLHS1 = % /. Solve[Q10eqn == 0, Q1,1][[1]]
(* check it *)
ν0d / x^5 * Sqrt[ΔΔ0 ΔΔm] * Q0d[1 / x] /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  Q0d[1 / x] → QQdkeval[9, 0, 1 / x];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
xnegLHS1 /. {XX1 → X1, XX2 → X2, XX3 → X3} /. {Q0d[1 / x] → QQdkeval[9, 0, 1 / x],
   Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0], Q2,0 → QQcxy[9, 2, 0]};
Simplify[%, Assumptions → t > 0 && x > 0];
% - %%% // Simplify
```

Out[•]= $4 a c \left(-\frac{1}{2} t^3 \left(-2 (-1+a) a^2 (-1+b)^2 + 2 b^2 \left(-(-1+a)^2 (-1+b) - a^2 (-b + a (-1+2 b)) t^3\right) - (-1+a) a^2 (-1+b) b^2 t^2 (XX_1 + XX_2)\right) \sqrt{XX_3}\, Q[0,0] - (-1+a) a^2 (-1+b) b^2 t^5 \sqrt{XX_3} \left(\frac{Q_{1,0} - a Q_{2,0} t}{a t} + x Q[0,0]\right) + \frac{1}{x^6} t (a (t-x) + x) (b (t-x) + x) \left(x - a (t+x) + a^2 t (1 + t x^2)\right) \left(x - b (t+x) + b^2 t (1 + t x^2)\right) \sqrt{(x - XX_1)(x - XX_2) XX_3}\, Q^d_0\left[\frac{1}{x}\right]\right)$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* then for the Q[x,0] part *)
μx,0 / x^5 / Sqrt[ΔΔp];
SeriesCoefficient[%, {x, 0, -5}] / x^5 *
   (Q[0, 0] + Q1,0 * x + Q2,0 * x^2 + Q3,0 * x^3 + Q4,0 * x^4) +
  SeriesCoefficient[%, {x, 0, -4}] / x^4 * (Q[0, 0] + Q1,0 * x + Q2,0 * x^2 + Q3,0 * x^3) +
  SeriesCoefficient[%, {x, 0, -3}] / x^3 * (Q[0, 0] + Q1,0 * x + Q2,0 * x^2) +
  SeriesCoefficient[%, {x, 0, -2}] / x^2 * (Q[0, 0] + Q1,0 * x) +
  SeriesCoefficient[%, {x, 0, -1}] / x * Q[0, 0];
xnegLHS2 = Collect[%, {Q[0, 0], Q1,0, Q2,0, Q3,0, Q4,0}]
(* check it *)
μx,0 / x^5 / Sqrt[ΔΔp] * Q[x, 0] /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  Q[x, 0] → QQcy[9, 0];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
xnegLHS2 /. {XX1 → X1, XX2 → X2, XX3 → X3} /.
  {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0],
   Q2,0 → QQcxy[9, 2, 0], Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0]};
% - %% // Simplify
```

Out[•]= $-\frac{2 a c Q_{4,0} t \left(-a t + a^2 t\right) (-a t - b t + 2 a b t) \left(-b t + b^2 t\right)}{x} +$

$$Q_{3,0}\left(-\frac{2\,a\,c\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{x^2}-\right.$$

$$\frac{1}{x}\,2\,c\,\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\left.\frac{a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{2\,XX_3}\Big)\right)+$$

$$Q_{1,0}\left(-\frac{2\,a\,c\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{x^4}-\right.$$

$$\frac{1}{x^3}\,2\,c\,\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\frac{a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{2\,XX_3}\Big)-$$

$$\frac{1}{x}\,2\,c\,\Big((1-b)\,\big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$(-b\,t+b^2\,t)\,\big((1-a)\,a^2\,t^2\,(-a\,t-b\,t+2\,a\,b\,t)+(2-a-b)\,((1-a)^2+a^3\,t^3)+$$

$$2\,a\,b\,t^2\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+b^2\,t^2\,\big(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+$$

$$(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\frac{5\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{16\,XX_3^3}+\frac{1}{8\,XX_3^2}$$

$$3\,\big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\big)+$$

$$\frac{1}{2\,XX_3}\,\big(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$

$$\big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+(1-b)\,\big(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\big)\Big)-$$

$$\frac{1}{x^2}\,2\,c\,\Big(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$

$$\big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+(1-b)\,\big(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\frac{3\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{8\,XX_3^2}+\frac{1}{2\,XX_3}$$

$$\big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t$$

$$\left.(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\big)\Big)\right)+$$

$$Q_{2,0}\left(-\frac{2\,a\,c\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{x^3}-\right.$$

$$\frac{1}{x^2}$$

$$2\,c\,\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+$$

$$(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+$$

$$(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\frac{a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{2\,XX_3}\Big)-$$

$$\frac{1}{x}\,2\,c\,\Big(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$

$$(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+(1-b)\,(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+$$

$$\frac{3\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{8\,XX_3^2}+\frac{1}{2\,XX_3}$$

$$(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))))\Big)+$$

$$\Big(-\frac{2\,a\,c\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{x^5}-$$

$$\frac{1}{x^4}$$

$$2\,c\,\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+$$

$$(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+$$

$$(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$

$$\frac{a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{2\,XX_3}\Big)-$$

$$\frac{1}{x^2}\,2\,c\,\Big((1-b)\,(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+$$

$$(-b\,t+b^2\,t)\,((1-a)\,a^2\,t^2\,(-a\,t-b\,t+2\,a\,b\,t)+(2-a-b)\,((1-a)^2+a^3\,t^3)+$$

$$2\,a\,b\,t^2\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+b^2\,t^2\,(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+$$

$$(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+$$

$$\frac{5\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{16\,XX_3^3}+\frac{1}{8\,XX_3^2}$$

$$3\,(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))))+$$

$$\frac{1}{2\,XX_3}\,(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$

$$(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$

$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))+(1-b)\,(a\,(2-a-b)\,t$$

$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)))))\Big)-$$

$$\frac{1}{x}\,2\,c\,\Big((-b\,t+b^2\,t)\,((1-a)\,a^2\,(2-a-b)\,t^2+2\,a\,b\,t^2\,((1-a)^2+a^3\,t^3))+$$

$$b^2\,t^2\,\Big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,\big((1-a)^2+a^3\,t^3\big)+(2-a-b)\,\big((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)\big)\Big)+$$
$$(1-b)\,\Big((1-a)\,a^2\,t^2\,(-a\,t-b\,t+2\,a\,b\,t)+(2-a-b)\,\big((1-a)^2+a^3\,t^3\big)+2\,a\,b\,t^2\,\big((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t)\big)\Big)+$$
$$\frac{35\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{128\,XX_3^4}+\frac{1}{16\,XX_3^3}$$
$$5\,\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\Big)+$$
$$\frac{1}{8\,XX_3^2}\,3\,\Big(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$
$$\big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+(2-a-b)$$
$$((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+(1-b)\,\big(a\,(2-a-b)\,t$$
$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\Big)+$$
$$\frac{1}{2\,XX_3}\,\Big((1-b)\,\big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$
$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+$$
$$(-b\,t+b^2\,t)\,\big((1-a)\,a^2\,t^2\,(-a\,t-b\,t+2\,a\,b\,t)+(2-a-b)\,((1-a)^2+a^3\,t^3)+$$
$$2\,a\,b\,t^2\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)+b^2\,t^2\,\big(a\,(2-a-b)\,t$$
$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\Big)\Bigg)-$$
$$\frac{1}{x^3}\,2\,c\,\Bigg(a\,b^2\,t^3\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+(-b\,t+b^2\,t)$$
$$\Big(2\,a^2\,b\,t^3\,(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)^2+a^3\,t^3)+$$
$$(2-a-b)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\Big)+(1-b)\,\Big(a\,(2-a-b)\,t$$
$$(-a\,t+a^2\,t)+(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\Big)+$$
$$\frac{3\,a\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)\,(-b\,t+b^2\,t)}{8\,XX_3^2}+\frac{1}{2\,XX_3}$$
$$\Big(a\,(1-b)\,t\,(-a\,t+a^2\,t)\,(-a\,t-b\,t+2\,a\,b\,t)+$$
$$(-b\,t+b^2\,t)\,\big(a\,(2-a-b)\,t\,(-a\,t+a^2\,t)+$$
$$(-a\,t-b\,t+2\,a\,b\,t)\,((1-a)\,a\,t+(1-a)\,(-a\,t+a^2\,t))\big)\Big)\Bigg)\,Q[0,0]$$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* then the Q1,0 part *)
μ1,0 / x^5 / Sqrt[ΔΔp];
(Series[%, {x, 0, -1}]) // Normal // Simplify;
ν1,0 / x^5 * Sqrt[ΔΔ0 ΔΔm];
(* the O[x,Infinity] term is to deal with some
 versions of Mathematica including unwanted things *)
% - (Normal[Series[%, {x, Infinity, 0}] + O[x, Infinity]^1]);
xnegRHS1 = (%%% + %) * Q1,0 // Simplify[#, Assumptions → t > 0 && x > 0] &
(* check it *)
Series[(μ1,0 + ν1,0 Sqrt[Δ]) / x^5 / Sqrt[Δp], {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS1 / Q1,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 9}];
Simplify[%, Assumptions → t > 0 && x > 0];
% - %%% // Simplify
```

Out[•]=

$$\frac{1}{8} c\, Q_{1,0}\, t \Bigg(32 (-1+a) a^2 (a-b) (-1+b) b t^3 \sqrt{XX_3} + \frac{1}{x^4} 16 (a-b) t \Big(2 (-1+b) b x^2 - 4 a (-1+b) b x^2 + a^2 \left(x^2 - 2 b^2 \left(-t x - x^2 + t^3 x^2 - t^2 (-1+x^3)\right) - b \left(2 t x + 3 x^2 + 2 t^2 (-1+x^3)\right)\right) + a^3 \left(-x^2 + 2 b^2 t \left(t - x + 2 t^2 x^2 - t x^3\right) + b \left(2 t x + x^2 - 2 t^3 x^2 + 2 t^2 (-1+x^3)\right)\right)\Big) \sqrt{(x - XX_1)(x - XX_2) XX_3} + \frac{1}{x^4 XX_3^3} \Big(8 (-1+b) b x^2 XX_3^2 \left(-4 x XX_3 + 2 b x XX_3 + b t (x + 2 XX_3)\right) - 8 a (-1+b) b x^2 XX_3^2 \left(2 (-5+2 b) x XX_3 + (-1+4 b) t (x + 2 XX_3)\right) - a^4 \Big(4 x^2 XX_3^2 \left(2 x XX_3 + t (x + 2 XX_3)\right) + b \left(-8 x^3 XX_3^3 - 20 t x^2 XX_3^2 (x + 2 XX_3) + 8 t^4 x^2 XX_3^2 (x + 2 XX_3) + t^3 \left(5 x^3 + 6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3\right)\right) + 2 b^3 t^2 \left(24 t^2 x^2 XX_3^2 (x + 2 XX_3) - 2 x XX_3 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right) + t \left(6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3 + x^3 (5 - 32 XX_3^3)\right)\right) + b^2 t \left(16 x^2 XX_3^2 (x + 2 XX_3) - 48 t^3 x^2 XX_3^2 (x + 2 XX_3) + 4 t x XX_3 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right) + 3 t^2 \left(-6 x^2 XX_3 - 8 x XX_3^2 - 16 XX_3^3 + x^3 (-5 + 16 XX_3^3)\right)\right)\Big) + a^2 \Big(-16 x^3 XX_3^3 + 4 b x XX_3 \left(22 x^2 XX_3^2 + 9 t x XX_3 (x + 2 XX_3) - t^2 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right)\right) - b^3 \left(-16 x^3 XX_3^3 - 48 t x^2 XX_3^2 (x + 2 XX_3) + 8 t^4 x^2 XX_3^2 (x + 2 XX_3) + t^3 \left(5 x^3 + 6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3\right)\right) + b^2 \left(-88 x^3 XX_3^3 - 84 t x^2 XX_3^2 (x + 2 XX_3) + 4 t^2 x XX_3 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right) + t^3 \left(6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3 + x^3 (5 + 16 XX_3^3)\right)\right)\Big) + a^3 \Big(4 x^2 XX_3^2 \left(6 x XX_3 + t (x + 2 XX_3)\right) - 4 b^2 \left(-6 x^3 XX_3^3 - 17 t x^2 XX_3^2 (x + 2 XX_3) + 8 t^4 x^2 XX_3^2 (x + 2 XX_3) + t^3 \left(5 x^3 + 6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3\right)\right) + b^3 t \left(-24 x^2 XX_3^2 (x + 2 XX_3) + 48 t^3 x^2 XX_3^2 (x + 2 XX_3) - 4 t x XX_3 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right) + 3 t^2 \left(6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3 + x^3 (5 - 16 XX_3^3)\right)\right) + b \left(-48 x^3 XX_3^3 - 48 t x^2 XX_3^2 (x + 2 XX_3) + 4 t^2 x XX_3 \left(3 x^2 + 4 x XX_3 + 8 XX_3^2\right) + t^3 \left(6 x^2 XX_3 + 8 x XX_3^2 + 16 XX_3^3 + x^3 (5 + 16 XX_3^3)\right)\right)\Big)\Big)\Bigg)$$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* then the Q2,0 part *)
μ2,0 / x^5 / Sqrt[ΔΔp];
(Series[%, {x, 0, -1}]) // Simplify // Normal;
ν2,0 / x^5 * Sqrt[ΔΔ0 ΔΔm];
(* the O[x,Infinity] term is to deal with some
 versions of Mathematica including unwanted things *)
% - (Normal[(Series[%, {x, Infinity, 0}] + O[x, Infinity]^1)]);
xnegRHS2 = (%%% + %) * Q2,0 // Simplify[#, Assumptions → t > 0 && x > 0] &
(* check it *)
Series[(μ2,0 + ν2,0 Sqrt[Δ]) / x^5 / Sqrt[Δp], {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS2 / Q2,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 9}];
Simplify[%, Assumptions → t > 0 && x > 0];
% - %%% // Simplify
```

Out[•]=
$$\frac{1}{2\,x^2\,XX_3}\,(-1+a)\,a^2\,(-1+b)\,(2+b)\,c\,Q_{2,0}\,t^2\left(4\,x\,XX_3-2\,a\,x\,XX_3-2\,b\,x\,XX_3-b\,t\left(x+XX_3\left(2-4\sqrt{(x-XX_1)\,(x-XX_2)\,XX_3}\right)\right)+a\,t\left((-1+2\,b)\,x-2\,XX_3\left(1-2\,b+2\sqrt{(x-XX_1)\,(x-XX_2)\,XX_3}\right)\right)\right)$$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* then the Q3,0 part *)
μ3,0 / x^5 / Sqrt[ΔΔp];
Series[%, {x, 0, -1}] // Simplify // Normal;
ν3,0 / x^5 * Sqrt[ΔΔ0 ΔΔm];
(* the O[x,Infinity] term is to deal with some
 versions of Mathematica including unwanted things *)
% - (Normal[(Series[%, {x, Infinity, 0}] + O[x, Infinity]^1)]);
xnegRHS3 = (%%% + %) * Q3,0 // Simplify[#, Assumptions → t > 0 && x > 0] &
(* check it *)
Series[(μ3,0 + ν3,0 Sqrt[Δ]) / x^5 / Sqrt[Δp], {t, 0, 9}];
ApplyToSeries[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &] &, %];
Series[xnegRHS3 / Q3,0 /. {XX1 → X1, XX2 → X2, XX3 → X3}, {t, 0, 9}];
Simplify[%, Assumptions → t > 0 && x > 0];
% - %%% // Simplify
```

Out[•]=
$$-\frac{1}{x^2\,XX_3}\,(-1+a)\,a^2\,(-1+b)\,b\,c\,Q_{3,0}\,t^3\left(4\,x\,XX_3-2\,a\,x\,XX_3-2\,b\,x\,XX_3-b\,t\left(x+XX_3\left(2-4\sqrt{(x-XX_1)\,(x-XX_2)\,XX_3}\right)\right)+a\,t\left((-1+2\,b)\,x-2\,XX_3\left(1-2\,b+2\sqrt{(x-XX_1)\,(x-XX_2)\,XX_3}\right)\right)\right)$$

Out[•]= $O[t]^{10}$

In[•]:=
```
(* now the constant term and the Q[0,0] coefficient are not so nice,
since they contain non-algebraic terms *)
(* in particular ν and ν0,0 cause trouble because they lead to
 (series with 1/x coefficients)*(series with x coefficients) *)
(* so the best we can do is give them a name or evaluate them manually *)
```

In[•]:=
```
(* the constant term *)
xnegRHS4s[N_] :=
 ApplyToSeries[Factor[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &]] &,
  μ / x^5 / Sqrt[Δp] + ν / x^5 * Sqrt[Δ0 Δm] /. θ → θs[N]]
```

In[•]:=
```
(* the Q[0,0] term *)
xnegRHS5s[N_] :=
 ApplyToSeries[Factor[Select[Expand[#] + x^π + x^(2 π), Exponent[#, x] < 0 &]] &,
   μ0,0 / x^5 / Sqrt[Δp] + ν0,0 / x^5 * Sqrt[Δ0 Δm] /. θ0,0 → θs0,0[N]] * Q[0, 0]
```

In[•]:=
```
xnegRHS4s[1]
xnegRHS5s[1]
```

Out[•]=
$$-\frac{2\left((-1+a)^2 a\,(-1+b)\,b\right)t}{x^2}+\frac{2\,(-1+a)\,a\,(-1+b)\,b\,\left(a^2-b+a\,b\right)t^2}{x^3}-\frac{2\left((-1+a)\,a\,b\,\left(-a^2+a^2 b^2+a\,x^3-a^2 x^3+b\,x^3-2\,a\,b\,x^3+a^2 b\,x^3\right)\right)t^3}{x^4}+O[t]^4$$

Out[•]=
$$\frac{2\,(-1+a)^3\,(-2+b)\,(-1+b)\,c\,Q[0,0]}{x}-\frac{2\left((-1+a)^2\,(-1+b)\,\left(-a\,b+a\,c-2\,a^2 c+3\,b\,c-3\,a\,b\,c+a^2 b\,c-b^2 c+a\,b^2 c\right)Q[0,0]\right)t}{x^2}+\frac{1}{x^3}2\,(-1+a)\,(-1+b)\,\left(-a^3 b+a\,b^2-a^2 b^2+2\,a^2 c-a^3 c-a\,b\,c+3\,a^2 b\,c-2\,a^3 b\,c-b^2 c+2\,a\,b^2 c-2\,a^2 b^2 c+a^3 b^2 c\right)Q[0,0]\,t^2+\frac{1}{x^4}\,2\,(-1+a)\,\left(-a^3 b+a^3 b^3+a^3 c+3\,a^2 b\,c-3\,a^3 b\,c-3\,a^2 b^2 c+2\,a^3 b^2 c+a^2 b\,x^3-a^3 b\,x^3+a\,b^2 x^3-2\,a^2 b^2 x^3+a^3 b^2 x^3+2\,a\,c\,x^3+a^2 c\,x^3+a^3 c\,x^3-6\,b\,c\,x^3+5\,a\,b\,c\,x^3-6\,a^2 b\,c\,x^3+a^3 b\,c\,x^3+8\,b^2 c\,x^3-11\,a\,b^2 c\,x^3+10\,a^2 b^2 c\,x^3-5\,a^3 b^2 c\,x^3-3\,b^3 c\,x^3+5\,a\,b^3 c\,x^3-4\,a^2 b^3 c\,x^3+2\,a^3 b^3 c\,x^3\right)Q[0,0]\,t^3+O[t]^4$$

In[•]:=
```
(* now setting up eqn (5.30) *)
P0d = (a^2 t^2 + x - a x - a t x^2 + a^2 t x^2) (b^2 t^2 + x - b x - b t x^2 + b^2 t x^2);
τ0d = 4 a c t (1 - a + a t x) (1 - b + b t x) Sqrt[(-1 + x XX1) (-1 + x XX2) XX3];
τ1,0 =
  Coefficient[(-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3) / x /. x → 1 / x,
   Q1,0] // Factor;
τ2,0 = Coefficient[(-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3) / x /.
    x → 1 / x, Q2,0] // Factor;
τ3,0 = Coefficient[(-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3) / x /.
    x → 1 / x, Q3,0] // Factor;
τ4,0 = Coefficient[(-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3) / x /.
    x → 1 / x, Q4,0] // Factor;
```

```
In[•]:= (* these are unwieldy and it will be more useful to have series expansions *)
(* first define these *)
Clear[τs0d, τs1,0, τs2,0, τs3,0, τs, τs0,0, xneg00]
(* then *)
τs0d[n_] := τs0d[n] = ApplyToSeries[Factor, Simplify[
    τ0d /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0 && x > 0]]
τs1,0[n_] := τs1,0[n] = ApplyToSeries[Factor, Simplify[
    τ1,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0 && x > 0]]
τs2,0[n_] := τs2,0[n] = ApplyToSeries[Factor, Simplify[
    τ2,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0 && x > 0]]
τs3,0[n_] := τs3,0[n] = ApplyToSeries[Factor, Simplify[
    τ3,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0 && x > 0]]
(* τ4,0 is a polynomial so we don't need to mess with it *)
(* these will also be useful *)
xneg00[n_] := xneg00[n] = ApplyToSeries[Factor, Simplify[Coefficient[
      (-xnegLHS1 - xnegLHS2 + xnegRHS1 + xnegRHS2 + xnegRHS3) / x /. x → 1 / x,
      Q[0, 0]] /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
(* then *)
τs[n_] := τs[n] = ApplyToSeries[Factor, Simplify[
    ((xnegRHS4s[n] / x /. x → 1 / x) /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} //
      Simplificate), Assumptions → t > 0 && x > 0]]
τs0,0[n_] := τs0,0[n] = ApplyToSeries[Factor, Simplify[
    ((xnegRHS5s[n] / Q[0, 0] / x /. x → 1 / x) + xneg00[n]) /. {XX1 → Xs1[n],
       XX2 → Xs2[n], XX3 → Xs3[n]} // Simplificate, Assumptions → t > 0 && x > 0]]

In[•]:= (* check it *)
-τs0d[9] P0d Q0d[x] + τs1,0[9] Q1,0 + τs2,0[9] Q2,0 + τs3,0[9] Q3,0 + τ4,0 Q4,0 + τs[9] +
   τs0,0[9] Q[0, 0] /. {Q0d[x] → QQdk[9, 0], Q1,0 → QQcxy[9, 1, 0], Q2,0 → QQcxy[9, 2, 0],
   Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0], Q[0, 0] → QQcxy[9, 0, 0]};
% // Simplificate;
ApplyToSeries[Factor, %]

Out[•]= O[t]^10
```

In[•]:=
```
(* the roots of P_{x,0} *)
```

$$x_1 = -\frac{1 - a + \sqrt{1 - 2 a + a^2 + 4 a^3 t^3 - 4 a^4 t^3}}{2 a^2 t^2};$$

$$x_2 = -\frac{1 - a - \sqrt{1 - 2 a + a^2 + 4 a^3 t^3 - 4 a^4 t^3}}{2 a^2 t^2};$$

$$x_3 = \frac{-1 + b - \sqrt{1 - 2 b + b^2 + 4 b^3 t^3 - 4 b^4 t^3}}{2 b^2 t^2};$$

$$x_4 = \frac{-1 + b + \sqrt{1 - 2 b + b^2 + 4 b^3 t^3 - 4 b^4 t^3}}{2 b^2 t^2};$$

$$x_5 = \frac{-2 + a + b - \sqrt{(2 - a - b)^2 - 8 a b t^2 (-a t - b t + 2 a b t)}}{4 a b t^2};$$

$$x_6 = \frac{-2 + a + b + \sqrt{(2 - a - b)^2 - 8 a b t^2 (-a t - b t + 2 a b t)}}{4 a b t^2};$$

```
(* so that *)
{P_{x,0} /. x → x_1, P_{x,0} /. x → x_2, P_{x,0} /. x → x_3,
  P_{x,0} /. x → x_4, P_{x,0} /. x → x_5, P_{x,0} /. x → x_6} // Simplify
```

Out[•]= {0, 0, 0, 0, 0, 0}

In[•]:=
```
(* and then verifying which are power series *)
Series[{x1, x2}, {t, 0, 4}];
Simplify[%, Assumptions → a > 1]
Simplify[%%, Assumptions → 0 < a < 1]
Series[{x3, x4}, {t, 0, 4}];
Simplify[%, Assumptions → b > 1]
Simplify[%%, Assumptions → 0 < b < 1]
Series[{x5, x6}, {t, 0, 4}];
Simplify[%, Assumptions → a + b > 2]
Simplify[%%, Assumptions → 0 < a + b < 2]
```

Out[•]= $\left\{a\,t + \frac{a^4\,t^4}{-1+a} + O[t]^5,\ \frac{-1+a}{a^2\,t^2} - a\,t + \frac{a^4\,t^4}{1-a} + O[t]^5\right\}$

Out[•]= $\left\{\frac{-1+a}{a^2\,t^2} - a\,t - \frac{a^4\,t^4}{-1+a} + O[t]^5,\ a\,t + \frac{a^4\,t^4}{-1+a} + O[t]^5\right\}$

Out[•]= $\left\{b\,t + \frac{b^4\,t^4}{-1+b} + O[t]^5,\ \frac{-1+b}{b^2\,t^2} - b\,t + \frac{b^4\,t^4}{1-b} + O[t]^5\right\}$

Out[•]= $\left\{\frac{-1+b}{b^2\,t^2} - b\,t - \frac{b^4\,t^4}{-1+b} + O[t]^5,\ b\,t + \frac{b^4\,t^4}{-1+b} + O[t]^5\right\}$

Out[•]= $\left\{\frac{(a+b-2\,a\,b)\,t}{2-a-b} + \frac{2\,a\,b\,(a+b-2\,a\,b)^2\,t^4}{(-2+a+b)^3} + O[t]^5,\right.$
$\left.\frac{-2+a+b}{2\,a\,b\,t^2} + \frac{(a+b-2\,a\,b)\,t}{-2+a+b} - \frac{2\left(a\,b\,(a+b-2\,a\,b)^2\right)\,t^4}{(-2+a+b)^3} + O[t]^5\right\}$

Out[•]= $\left\{\frac{-2+a+b}{2\,a\,b\,t^2} + \frac{(a+b-2\,a\,b)\,t}{-2+a+b} - \frac{2\left(a\,b\,(a+b-2\,a\,b)^2\right)\,t^4}{(-2+a+b)^3} + O[t]^5,\right.$
$\left.-\frac{(a+b-2\,a\,b)\,t}{-2+a+b} + \frac{2\,a\,b\,(a+b-2\,a\,b)^2\,t^4}{(-2+a+b)^3} + O[t]^5\right\}$

In[•]:= (* the roots of P_0^d *)

$x_7 = \frac{-1+a-\sqrt{1-2\,a+a^2+4\,a^3\,t^3-4\,a^4\,t^3}}{2\,t\,a\,(a-1)};$

$x_8 = \frac{-1+a+\sqrt{1-2\,a+a^2+4\,a^3\,t^3-4\,a^4\,t^3}}{2\,t\,a\,(a-1)};$

$x_9 = \frac{-1+b-\sqrt{1-2\,b+b^2+4\,b^3\,t^3-4\,b^4\,t^3}}{2\,t\,b\,(b-1)};$

$x_{10} = \frac{-1+b+\sqrt{1-2\,b+b^2+4\,b^3\,t^3-4\,b^4\,t^3}}{2\,t\,b\,(b-1)};$

(* so that *)

$\{P_0^d$ /. x → x_7, P_0^d /. x → x_8, P_0^d /. x → x_9, P_0^d /. x → $x_{10}\}$ // Simplify

Out[•]= {0, 0, 0, 0}

In[•]:=
```
(* and then verifying which are power series *)
Series[{x7, x8}, {t, 0, 5}];
Simplify[%, Assumptions → a > 1]
Simplify[%%, Assumptions → 0 < a < 1]
Series[{x9, x10}, {t, 0, 5}];
Simplify[%, Assumptions → b > 1]
Simplify[%%, Assumptions → 0 < b < 1]
```

Out[•]= $$\left\{\frac{a^2 t^2}{-1+a}+\frac{a^5 t^5}{(-1+a)^2}+O[t]^6,\ \frac{1}{a\,t}+\frac{a^2 t^2}{1-a}-\frac{a^5 t^5}{(-1+a)^2}+O[t]^6\right\}$$

Out[•]= $$\left\{\frac{1}{a\,t}-\frac{a^2 t^2}{-1+a}-\frac{a^5 t^5}{(-1+a)^2}+O[t]^6,\ \frac{a^2 t^2}{-1+a}+\frac{a^5 t^5}{(-1+a)^2}+O[t]^6\right\}$$

Out[•]= $$\left\{\frac{b^2 t^2}{-1+b}+\frac{b^5 t^5}{(-1+b)^2}+O[t]^6,\ \frac{1}{b\,t}+\frac{b^2 t^2}{1-b}-\frac{b^5 t^5}{(-1+b)^2}+O[t]^6\right\}$$

Out[•]= $$\left\{\frac{1}{b\,t}-\frac{b^2 t^2}{-1+b}-\frac{b^5 t^5}{(-1+b)^2}+O[t]^6,\ \frac{b^2 t^2}{-1+b}+\frac{b^5 t^5}{(-1+b)^2}+O[t]^6\right\}$$

In[•]:=
```
(* these will be useful *)
Clear[xs1, xs2, xs3, xs4, xs5, xs6, xs7, xs8, xs9, xs10]
xs1[n_] := xs1[n] =
  ApplyToSeries[Factor[Simplify[#, Assumptions → a > 1]] &, Series[x1, {t, 0, n}]]
xs2[n_] := xs2[n] = ApplyToSeries[
   Factor[Simplify[#, Assumptions → 0 < a < 1]] &, Series[x2, {t, 0, n}]]
xs3[n_] := xs3[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → b > 1]] &,
   Series[x3, {t, 0, n}]]
xs4[n_] := xs4[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < b < 1]] &,
   Series[x4, {t, 0, n}]]
xs5[n_] := xs5[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → a + b > 2]] &,
   Series[x5, {t, 0, n}]]
xs6[n_] := xs6[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < a + b < 2]] &,
   Series[x6, {t, 0, n}]]
xs7[n_] := xs7[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → a > 1]] &,
   Series[x7, {t, 0, n}]]
xs8[n_] := xs8[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < a < 1]] &,
   Series[x8, {t, 0, n}]]
xs9[n_] := xs9[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → b > 1]] &,
   Series[x9, {t, 0, n}]]
xs10[n_] := xs10[n] = ApplyToSeries[Factor[Simplify[#, Assumptions → 0 < b < 1]] &,
   Series[x10, {t, 0, n}]]
```

```
In[•]:= (* now verifying what happens when we cancel the kernel *)
        {σs[9], σs0,0[9], σs1,0[9], σs2,0[9], σs3,0[9], σ4,0}.
          {1, Q[0, 0], Q1,0, Q2,0, Q3,0, Q4,0};
        % /. {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0],
            Q2,0 → QQcxy[9, 2, 0], Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0]};
        % // Simplificate;
        {% /. x → xs1[9], % /. x → xs3[9], % /. x → xs5[9]};
        Simplificate /@ %
        {τs[9], τs0,0[9], τs1,0[9], τs2,0[9], τs3,0[9], τ4,0}.
          {1, Q[0, 0], Q1,0, Q2,0, Q3,0, Q4,0};
        % /. {Q[0, 0] → QQcxy[9, 0, 0], Q1,0 → QQcxy[9, 1, 0],
            Q2,0 → QQcxy[9, 2, 0], Q3,0 → QQcxy[9, 3, 0], Q4,0 → QQcxy[9, 4, 0]};
        % // Simplificate;
        {% /. x → xs7[9], % /. x → xs9[9]};
        Simplificate /@ %
```

Out[•]= $\{O[t]^{10}, O[t]^{10}, O[t]^{10}\}$

Out[•]= $\{O[t]^{10}, O[t]^{10}\}$

```
In[•]:= (* now we have 5 equations with 5 unknowns *)
        (* but does this lead to a solution? *)
        (* form the 5x5 matrix of coefficients and find the determinant *)
        xposcoeffs[n_] := {σs0,0[n], σs1,0[n], σs2,0[n], σs3,0[n], σ4,0};
        xnegcoeffs[n_] := {τs0,0[n], τs1,0[n], τs2,0[n], τs3,0[n], τ4,0};
        {xposcoeffs[12] /. x → xs1[12],
          xposcoeffs[12] /. x → xs3[12], xposcoeffs[12] /. x → xs5[12],
          xnegcoeffs[12] /. x → xs7[12], xnegcoeffs[12] /. x → xs9[12]};
        Simplificate /@ # & /@ %;
        Det[%]
```

Out[•]= $O[t]^{40}$

```
In[•]:= (* it appears not *)
```

```
In[•]:= (* we can introduce a sixth equation
         by taking the [x^0] part of eqn (5.20) *)
        μx,0 / x^4 / Sqrt[ΔΔp];
        x0LHS1 =
         SeriesCoefficient[%, {x, 0, -4}] * Q4,0 + SeriesCoefficient[%, {x, 0, -3}] * Q3,0 +
           SeriesCoefficient[%, {x, 0, -2}] * Q2,0 + SeriesCoefficient[%, {x, 0, -1}] * Q1,0 +
           SeriesCoefficient[%, {x, 0, 0}] * Q[0, 0] // Simplify
```

Out[•]= $-\frac{1}{64\, XX_3^4} c \Big(a^3 \big(8\, XX_3 \big(16\, Q_{3,0}\, t^3\, XX_3^3 + 8\, Q_{2,0}\, t^2\, XX_3^2 (t + 4\, XX_3) + 2\, Q_{1,0}\, t\, XX_3 (3\, t^2 + 8\, t\, XX_3 - 32\, XX_3^2) +$
$5\, t^3\, Q[0, 0] + 12\, t^2\, XX_3\, Q[0, 0] - 32\, t\, XX_3^2\, Q[0, 0] - 16\, XX_3^3\, Q[0, 0] +$
$16\, t^3\, XX_3^3\, Q[0, 0]\big) + 4\, b^2\, t \big(64\, Q_{3,0}\, t^3\, XX_3^3 + 96\, Q_{3,0}\, t^2\, XX_3^4 +$
$128\, Q_{4,0}\, t^3\, XX_3^4 + 16\, Q_{2,0}\, t\, XX_3^2 (3\, t^2 + 3\, t\, XX_3 - 18\, XX_3^2 + 8\, t^3\, XX_3^2) +$
$4\, Q_{1,0}\, XX_3 \big(9\, t^2\, XX_3 - 36\, t\, XX_3^2 + 16\, t^4\, XX_3^2 - 32\, XX_3^3 + 2\, t^3 (5 + 8\, XX_3^3)\big) +$
$35\, t^3\, Q[0, 0] + 30\, t^2\, XX_3\, Q[0, 0] - 108\, t\, XX_3^2\, Q[0, 0] + 48\, t^4\, XX_3^2\, Q[0, 0] -$

$$64\, XX_3^3\, Q[0,0] + 32\, t^3\, XX_3^3\, Q[0,0] - 448\, t^2\, XX_3^4\, Q[0,0]) - b\, (64\, Q_{3,0}\, t^4\, XX_3^3 +$$
$$768\, Q_{3,0}\, t^3\, XX_3^4 + 128\, Q_{4,0}\, t^4\, XX_3^4 + 16\, Q_{2,0}\, t^2\, XX_3^2\, (3\, t^2 + 24\, t\, XX_3 - 32\, XX_3^2) +$$
$$8\, Q_{1,0}\, t\, XX_3\, (36\, t^2\, XX_3 - 32\, t\, XX_3^2 - 128\, XX_3^3 + t^3\, (5 + 64\, XX_3^3)) + 35\, t^4\, Q[0,0] +$$
$$240\, t^3\, XX_3\, Q[0,0] - 192\, t^2\, XX_3^2\, Q[0,0] - 512\, t\, XX_3^3\, Q[0,0] +$$
$$256\, t^4\, XX_3^3\, Q[0,0] - 128\, XX_3^4\, Q[0,0] - 768\, t^3\, XX_3^4\, Q[0,0]) +$$
$$b^3\, t^2\, (-192\, Q_{3,0}\, t^2\, XX_3^3 + 256\, Q_{3,0}\, t\, XX_3^4 - 384\, Q_{4,0}\, t^2\, XX_3^4 -$$
$$16\, Q_{2,0}\, XX_3^2\, (9\, t^2 - 8\, t\, XX_3 - 24\, XX_3^2 + 48\, t^3\, XX_3^2) +$$
$$24\, Q_{1,0}\, XX_3\, (4\, t\, XX_3 + 8\, XX_3^2 - 16\, t^3\, XX_3^2 + t^2\, (-5 + 32\, XX_3^3)) -$$
$$105\, t^2\, Q[0,0] + 80\, t\, XX_3\, Q[0,0] + 144\, XX_3^2\, Q[0,0] - 288\, t^3\, XX_3^2\, Q[0,0] +$$
$$384\, t^2\, XX_3^3\, Q[0,0] + 768\, t\, XX_3^4\, Q[0,0] - 384\, t^4\, XX_3^4\, Q[0,0])) +$$
$$a^4\, t\, (-8\, XX_3\, (8\, Q_{2,0}\, t^2\, XX_3^2 + 16\, Q_{3,0}\, t^2\, XX_3^3 + 2\, Q_{1,0}\, (3\, t^2\, XX_3 - 8\, XX_3^3 + 8\, t^3\, XX_3^3) +$$
$$5\, t^2\, Q[0,0] - 8\, XX_3^2\, Q[0,0] + 8\, t^3\, XX_3^2\, Q[0,0]) +$$
$$b\, (64\, Q_{3,0}\, t^3\, XX_3^3 + 384\, Q_{3,0}\, t^2\, XX_3^4 + 128\, Q_{4,0}\, t^3\, XX_3^4 +$$
$$16\, Q_{2,0}\, t\, XX_3^2\, (3\, t^2 + 12\, t\, XX_3 - 24\, XX_3^2 + 8\, t^3\, XX_3^2) +$$
$$8\, Q_{1,0}\, XX_3\, (18\, t^2\, XX_3 - 24\, t\, XX_3^2 + 8\, t^4\, XX_3^2 - 16\, XX_3^3 + t^3\, (5 + 80\, XX_3^3)) +$$
$$35\, t^3\, Q[0,0] + 120\, t^2\, XX_3\, Q[0,0] - 144\, t\, XX_3^2\, Q[0,0] + 48\, t^4\, XX_3^2\, Q[0,0] -$$
$$64\, XX_3^3\, Q[0,0] + 320\, t^3\, XX_3^3\, Q[0,0] - 640\, t^2\, XX_3^4\, Q[0,0]) + 2\, b^3\, t^2$$
$$(64\, Q_{3,0}\, t\, XX_3^3 - 128\, Q_{3,0}\, XX_3^4 + 128\, Q_{4,0}\, t\, XX_3^4 + 16\, Q_{2,0}\, XX_3^2\, (3\, t - 4\, XX_3 + 24\, t^2\, XX_3^2) +$$
$$8\, Q_{1,0}\, XX_3\, (-6\, XX_3 + 24\, t^2\, XX_3^2 + t\, (5 - 48\, XX_3^3)) + 35\, t\, Q[0,0] -$$
$$40\, XX_3\, Q[0,0] + 144\, t^2\, XX_3^2\, Q[0,0] - 192\, t\, XX_3^3\, Q[0,0] + 384\, t^3\, XX_3^4\, Q[0,0]) -$$
$$3\, b^2\, t\, (64\, Q_{3,0}\, t^2\, XX_3^3 + 128\, Q_{4,0}\, t^2\, XX_3^4 + 8\, Q_{1,0}\, (5\, t^2\, XX_3 - 8\, XX_3^3 + 16\, t^3\, XX_3^3) +$$
$$16\, Q_{2,0}\, (3\, t^2\, XX_3^2 - 8\, XX_3^4 + 16\, t^3\, XX_3^4) + 35\, t^2\, Q[0,0] - 48\, XX_3^2\, Q[0,0] +$$
$$96\, t^3\, XX_3^2\, Q[0,0] - 256\, t\, XX_3^4\, Q[0,0] + 128\, t^4\, XX_3^4\, Q[0,0])) +$$
$$a^2\, (-8\, b\, XX_3\, (-8\, Q_{2,0}\, t^2\, (3\, t - 4\, XX_3)\, XX_3^2 - 48\, Q_{3,0}\, t^3\, XX_3^3 + 2\, Q_{1,0}\, t\, XX_3$$
$$(-9\, t^2 + 8\, t\, XX_3 + 112\, XX_3^2) - 15\, t^3\, Q[0,0] + 12\, t^2\, XX_3\, Q[0,0] +$$
$$112\, t\, XX_3^2\, Q[0,0] + 80\, XX_3^3\, Q[0,0] + 48\, t^3\, XX_3^3\, Q[0,0]) +$$
$$b^3\, t\, (64\, Q_{3,0}\, t^3\, XX_3^3 + 128\, Q_{4,0}\, t^3\, XX_3^4 + 16\, Q_{2,0}\, (3\, t^3\, XX_3^2 - 48\, t\, XX_3^4 + 8\, t^4\, XX_3^4) +$$
$$8\, Q_{1,0}\, (-48\, t\, XX_3^3 + 8\, t^4\, XX_3^3 - 16\, XX_3^4 + t^3\, (5\, XX_3 - 16\, XX_3^4)) +$$
$$35\, t^3\, Q[0,0] - 288\, t\, XX_3^2\, Q[0,0] + 48\, t^4\, XX_3^2\, Q[0,0] - 64\, XX_3^3\, Q[0,0] -$$
$$64\, t^3\, XX_3^3\, Q[0,0] - 1408\, t^2\, XX_3^4\, Q[0,0]) - b^2\, (64\, Q_{3,0}\, t^4\, XX_3^3 +$$
$$384\, Q_{3,0}\, t^3\, XX_3^4 + 128\, Q_{4,0}\, t^4\, XX_3^4 + 16\, Q_{2,0}\, t^2\, XX_3^2\, (3\, t^2 + 12\, t\, XX_3 - 80\, XX_3^2) +$$
$$8\, Q_{1,0}\, t\, XX_3\, (18\, t^2\, XX_3 - 80\, t\, XX_3^2 - 176\, XX_3^3 + t^3\, (5 + 16\, XX_3^3)) +$$
$$35\, t^4\, Q[0,0] + 120\, t^3\, XX_3\, Q[0,0] - 480\, t^2\, XX_3^2\, Q[0,0] - 704\, t\, XX_3^3\, Q[0,0] +$$
$$64\, t^4\, XX_3^3\, Q[0,0] - 128\, XX_3^4\, Q[0,0] - 1536\, t^3\, XX_3^4\, Q[0,0]) + 32\, XX_3^2$$
$$(-4\, Q_{1,0}\, t\, (t - 4\, XX_3)\, XX_3 - 8\, Q_{2,0}\, t^2\, XX_3^2 + (-3\, t^2 + 8\, t\, XX_3 + 16\, XX_3^2)\, Q[0,0])) -$$
$$16\, XX_3^2\, (-16\, XX_3^2\, Q[0,0] + 12\, b\, XX_3\, (2\, Q_{1,0}\, t\, XX_3 + (t + 2\, XX_3)\, Q[0,0]) -$$
$$b^2\, (8\, Q_{2,0}\, t^2\, XX_3^2 + 4\, Q_{1,0}\, t\, XX_3\, (t + 8\, XX_3) + (3\, t^2 + 16\, t\, XX_3 + 8\, XX_3^2)\, Q[0,0]) +$$
$$b^3\, t\, (8\, Q_{2,0}\, t\, XX_3^2 + 4\, Q_{1,0}\, XX_3\, (t + 2\, XX_3) + (3\, t + 4\, XX_3 + 8\, t^2\, XX_3^2)\, Q[0,0])) +$$
$$16\, a\, XX_3^2\, (-4\, XX_3\, (2\, Q_{1,0}\, t\, XX_3 + (t + 10\, XX_3)\, Q[0,0]) +$$
$$b\, (8\, Q_{2,0}\, t^2\, XX_3^2 + 4\, Q_{1,0}\, t\, XX_3\, (t + 20\, XX_3) + (3\, t^2 + 40\, t\, XX_3 + 56\, XX_3^2)\, Q[0,0]) +$$
$$4\, b^3\, t\, (8\, Q_{2,0}\, t\, XX_3^2 + 4\, Q_{1,0}\, XX_3\, (t + XX_3) + (3\, t + 2\, XX_3 + 12\, t^2\, XX_3^2)\, Q[0,0]) -$$
$$b^2\, (40\, Q_{2,0}\, t^2\, XX_3^2 + 4\, Q_{1,0}\, t\, XX_3\, (5\, t + 22\, XX_3) +$$
$$(15\, t^2 + 44\, t\, XX_3 + 16\, XX_3^2 + 24\, t^3\, XX_3^2)\, Q[0,0])))$$

In[•]:=
```
ν_0^d / x^4 * Sqrt[ΔΔ_0 ΔΔ_m];
SeriesCoefficient[%, {x, Infinity, -2}] * Q_{2,2} +
   SeriesCoefficient[%, {x, Infinity, -1}] * Q_{1,1} +
   SeriesCoefficient[%, {x, Infinity, 0}] * Q[0, 0] //
  Simplify[#, Assumptions → t > 0] &;
% /. Solve[Q21eqn == 0, Q_{2,2}][[1]] /. Solve[Q10eqn == 0, Q_{1,1}][[1]] /.
   Solve[Q20eqn == 0, Q_{2,1}][[1]] /. Solve[Q30eqn == 0, Q_{3,1}][[1]];
x0LHS2 = % // Simplify
```

Out[•]=

$$\frac{1}{2} a c t \sqrt{XX_3} \Big(-8 (-1+a) a (-1+b) b^2 t^2 \left(-Q_{2,0}+t \left((1+a) Q_{3,0}+a \left(Q_{1,0}-Q_{4,0}\right) t\right)\right) - \frac{1}{a} 4 t \left(Q_{1,0}-a Q_{2,0} t\right) \Big(2 (-1+a) a^2 (-1+b)^2 + 2 (-1+a)^2 (-1+b) b^2 + 2 a^2 b^2 \left(-b+a (-1+2 b)\right) t^3 + (-1+a) a^2 (-1+b) b^2 t^2 \left(XX_1+XX_2\right)\Big) + \Big(8 \Big(1-2 b+b^2+b^3 t^3-2 a \left(1-2 b+b^2+b^3 t^3\right)+a^3 t^3 \left(1-2 b-b^2+b^3 \left(2+t^3\right)\right) + a^2 \left(1-2 b-b^3 t^3+b^2 \left(1+2 t^3\right)\right)\Big) - (-1+a) a^2 (-1+b) b^2 t^4 \left(XX_1-XX_2\right)^2 - 4 t^2 \left(-(-1+a) a^2 (-1+b)^2+b^2 \left(-(-1+a)^2 (-1+b)-a^2 \left(-b+a (-1+2 b)\right) t^3\right)\right) \left(XX_1+XX_2\right)\Big) Q[0,0]\Big)$$

In[•]:=
```
SeriesCoefficient[μ_{1,0} / x^4 / Sqrt[ΔΔ_p], {x, 0, 0}] +
  SeriesCoefficient[ν_{1,0} / x^4 * Sqrt[ΔΔ_0 ΔΔ_m], {x, Infinity, 0}];
x0RHS1 = Simplify[%, Assumptions → t > 0] * Q_{1,0}
```

Out[•]=

$$-\frac{1}{8 XX_3^3} c Q_{1,0} t \Big(8 (-1+b) b XX_3^2 \left(4 XX_3-2 b XX_3+b t \left(-1+4 XX_3^{3/2}\right)\right) - 8 a (-1+b) b XX_3^2 \left(t+2 (5-2 b) XX_3+4 t XX_3^{3/2}+b t \left(-4+8 XX_3^{3/2}\right)\right) + a^4 \Big(4 XX_3^2 \left(t+2 XX_3+4 t XX_3^{3/2}\right)+2 b^3 t^2 \left(-6 XX_3+24 t^2 XX_3^2+t \left(5-32 XX_3^3\right)\right) + b \left(-8 XX_3^3-4 t XX_3^2 \left(5+4 XX_3^{3/2}\right)+8 t^4 \left(XX_3^2+4 XX_3^{7/2}\right) + t^3 \left(5+16 XX_1 XX_3^{7/2}+16 XX_2 XX_3^{7/2}\right)\right) - b^2 t \left(-12 t XX_3-16 XX_3^2 + 16 t^3 XX_3^2 \left(3+4 XX_3^{3/2}\right)+t^2 \left(15-48 XX_3^3+16 \left(XX_1+XX_2\right) XX_3^{7/2}\right)\right)\Big) + a^2 \Big(16 XX_3^3-4 b XX_3 \left(-3 t^2+22 XX_3^2+3 t XX_3 \left(3+4 XX_3^{3/2}\right)\right) - b^3 \left(16 XX_3^3+8 t^4 XX_3^2 \left(-1+4 XX_3^{3/2}\right)+t \left(48 XX_3^2-32 XX_3^{7/2}\right) + t^3 \left(-5+16 XX_1 XX_3^{7/2}+16 XX_2 XX_3^{7/2}\right)\right)+b^2 \left(-12 t^2 XX_3+88 XX_3^3 + 4 t XX_3^2 \left(21+4 XX_3^{3/2}\right)+t^3 \left(-5-16 XX_3^3+16 \left(XX_1+XX_2\right) XX_3^{7/2}\right)\right)\Big) + a^3 \Big(-4 XX_3^2 \left(t+6 XX_3+4 t XX_3^{3/2}\right)+4 b^2 \left(5 t^3+8 t^4 XX_3^2-6 XX_3^3-t XX_3^2 \left(17+4 XX_3^{3/2}\right)\right) - b \left(12 t^2 XX_3-48 XX_3^3-16 t XX_3^2 \left(3+2 XX_3^{3/2}\right)+t^3 \left(5+16 XX_3^3+16 \left(XX_1+XX_2\right) XX_3^{7/2}\right)\right) + b^3 t \left(12 t XX_3+24 XX_3^2+16 t^3 XX_3^2 \left(-3+4 XX_3^{3/2}\right) + t^2 \left(-15+48 XX_3^3+16 \left(XX_1+XX_2\right) XX_3^{7/2}\right)\right)\Big)\Big)$$

In[•]:=
```
SeriesCoefficient[μ_{2,0} / x^4 / Sqrt[ΔΔ_p], {x, 0, 0}] +
  SeriesCoefficient[ν_{2,0} / x^4 * Sqrt[ΔΔ_0 ΔΔ_m], {x, Infinity, 0}];
x0RHS2 = Simplify[%, Assumptions → t > 0] * Q_{2,0}
```

Out[•]=

$$\frac{1}{2 XX_3} (-1+a) a^2 (-1+b) (2+b) c Q_{2,0} t^2 \left(4 XX_3+b \left(-t-2 XX_3+4 t XX_3^{3/2}\right)-a \left(t-2 b t+2 XX_3+4 t XX_3^{3/2}\right)\right)$$

```
In[•]:= SeriesCoefficient[μ3,0 / x^4 / Sqrt[ΔΔp], {x, 0, 0}] +
          SeriesCoefficient[ν3,0 / x^4 * Sqrt[ΔΔ0 ΔΔm], {x, Infinity, 0}];
        x0RHS3 = Simplify[%, Assumptions → t > 0] * Q3,0
```

Out[•]= $-\frac{1}{XX_3}(-1+a)\,a^2\,(-1+b)\,b\,c\,Q_{3,0}\,t^3$
$\left(4\,XX_3+b\,\left(-t-2\,XX_3+4\,t\,XX_3^{3/2}\right)-a\,\left(t-2\,b\,t+2\,XX_3+4\,t\,XX_3^{3/2}\right)\right)$

```
In[•]:= x0RHS4s[N_] := ApplyToSeries[Factor[Coefficient[Expand[#], x, 0]] &,
          μ / x^4 / Sqrt[Δp] + ν / x^4 * Sqrt[Δ0 Δm] /. θ → θs[N]]
```

```
In[•]:= x0RHS5s[N_] := ApplyToSeries[Factor[Coefficient[Expand[#], x, 0]] &,
          μ0,0 / x^4 / Sqrt[Δp] + ν0,0 / x^4 * Sqrt[Δ0 Δm] /. θ0,0 → θs0,0[N]] * Q[0, 0]
```

```
In[•]:= Clear[ξ0s1,0, ξ0s2,0, ξ0s3,0, ξ0s4,0, x000, ξ0s, ξ0s0,0]
        ξ01,0 =
         Coefficient[-x0LHS1 - x0LHS2 + x0RHS1 + x0RHS2 + x0RHS3, Q1,0] // Simplify // Factor
        ξ02,0 = Coefficient[-x0LHS1 - x0LHS2 + x0RHS1 + x0RHS2 + x0RHS3, Q2,0] // Simplify //
          Factor
        ξ03,0 = Coefficient[-x0LHS1 - x0LHS2 + x0RHS1 + x0RHS2 + x0RHS3, Q3,0] // Simplify //
          Factor
        ξ04,0 = Coefficient[-x0LHS1 - x0LHS2 + x0RHS1 + x0RHS2 + x0RHS3, Q4,0] // Simplify //
          Factor
        ξ0s1,0[n_] := ξ0s1,0[n] = ApplyToSeries[Factor,
           Simplify[ξ01,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
        ξ0s2,0[n_] := ξ0s2,0[n] = ApplyToSeries[Factor,
           Simplify[ξ02,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
        ξ0s3,0[n_] := ξ0s3,0[n] = ApplyToSeries[Factor,
           Simplify[ξ03,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
        ξ0s4,0[n_] := ξ0s4,0[n] = ApplyToSeries[Factor,
           Simplify[ξ04,0 /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
        x000[n_] := x000[n] = ApplyToSeries[Factor,
           Simplify[Coefficient[-x0LHS1 - x0LHS2 + x0RHS1 + x0RHS2 + x0RHS3, Q[0, 0]] /.
              {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]}, Assumptions → t > 0]]
        ξ0s[n_] := ξ0s[n] = ApplyToSeries[Factor, Simplify[
             (x0RHS4s[n] /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} // Simplificate),
             Assumptions → t > 0]]
        ξ0s0,0[n_] := ξ0s0,0[n] = ApplyToSeries[Factor, Simplify[
             (x0RHS5s[n] / Q[0, 0] + x000[n]) /. {XX1 → Xs1[n], XX2 → Xs2[n], XX3 → Xs3[n]} //
              Simplificate, Assumptions → t > 0]]
```

Out[•]=
$$\begin{aligned}&\frac{1}{4\,XX_3^2}\,c\,t\,\big(3\,a^3\,t^2-3\,a^4\,t^2+3\,a^2\,b\,t^2-12\,a^3\,b\,t^2+9\,a^4\,b\,t^2-3\,a^2\,b^2\,t^2+9\,a^3\,b^2\,t^2-6\,a^4\,b^2\,t^2-\\
&8\,a^2\,t\,XX_3+10\,a^3\,t\,XX_3-2\,a^4\,t\,XX_3+10\,a^2\,b\,t\,XX_3-8\,a^3\,b\,t\,XX_3-2\,a^4\,b\,t\,XX_3-\\
&2\,a^2\,b^2\,t\,XX_3-2\,a^3\,b^2\,t\,XX_3+4\,a^4\,b^2\,t\,XX_3-8\,a\,XX_3^2+24\,a^2\,XX_3^2-20\,a^3\,XX_3^2+\\
&4\,a^4\,XX_3^2-8\,b\,XX_3^2+40\,a\,b\,XX_3^2-68\,a^2\,b\,XX_3^2+40\,a^3\,b\,XX_3^2-4\,a^4\,b\,XX_3^2+8\,b^2\,XX_3^2-\\
&32\,a\,b^2\,XX_3^2+44\,a^2\,b^2\,XX_3^2-20\,a^3\,b^2\,XX_3^2-8\,a^4\,t^3\,XX_3^2-24\,a^3\,b\,t^3\,XX_3^2+40\,a^4\,b\,t^3\,XX_3^2+\\
&16\,a^3\,b^2\,t^3\,XX_3^2-24\,a^4\,b^2\,t^3\,XX_3^2-8\,a^2\,b^3\,t^3\,XX_3^2+24\,a^3\,b^3\,t^3\,XX_3^2-16\,a^4\,b^3\,t^3\,XX_3^2-\\
&16\,a^2\,t\,XX_3^{5/2}+24\,a^3\,t\,XX_3^{5/2}-8\,a^4\,t\,XX_3^{5/2}-16\,a\,b\,t\,XX_3^{5/2}+56\,a^2\,b\,t\,XX_3^{5/2}-\\
&48\,a^3\,b\,t\,XX_3^{5/2}+8\,a^4\,b\,t\,XX_3^{5/2}+16\,a\,b^2\,t\,XX_3^{5/2}-40\,a^2\,b^2\,t\,XX_3^{5/2}+24\,a^3\,b^2\,t\,XX_3^{5/2}-\\
&16\,a^4\,b\,t^4\,XX_3^{5/2}+16\,a^4\,b^2\,t^4\,XX_3^{5/2}-16\,a^3\,b^3\,t^4\,XX_3^{5/2}+16\,a^4\,b^3\,t^4\,XX_3^{5/2}+\\
&8\,a^3\,b\,t^3\,XX_1\,XX_3^{5/2}-8\,a^4\,b\,t^3\,XX_1\,XX_3^{5/2}-8\,a^3\,b^2\,t^3\,XX_1\,XX_3^{5/2}+8\,a^4\,b^2\,t^3\,XX_1\,XX_3^{5/2}+\\
&8\,a^3\,b\,t^3\,XX_2\,XX_3^{5/2}-8\,a^4\,b\,t^3\,XX_2\,XX_3^{5/2}-8\,a^3\,b^2\,t^3\,XX_2\,XX_3^{5/2}+8\,a^4\,b^2\,t^3\,XX_2\,XX_3^{5/2}\big)\end{aligned}$$

Out[•]=
$$\begin{aligned}&-\frac{1}{4\,XX_3^2}\,c\,t^2\\
&\big(3\,a^3\,b\,t^2-3\,a^4\,b\,t^2+3\,a^2\,b^2\,t^2-12\,a^3\,b^2\,t^2+9\,a^4\,b^2\,t^2-3\,a^2\,b^3\,t^2+9\,a^3\,b^3\,t^2-6\,a^4\,b^3\,t^2-\\
&8\,a^2\,b\,t\,XX_3+10\,a^3\,b\,t\,XX_3-2\,a^4\,b\,t\,XX_3+10\,a^2\,b^2\,t\,XX_3-8\,a^3\,b^2\,t\,XX_3-2\,a^4\,b^2\,t\,XX_3-\\
&2\,a^2\,b^3\,t\,XX_3-2\,a^3\,b^3\,t\,XX_3+4\,a^4\,b^3\,t\,XX_3+8\,a^3\,XX_3^2-8\,a^4\,XX_3^2-8\,a\,b\,XX_3^2+\\
&32\,a^2\,b\,XX_3^2-52\,a^3\,b\,XX_3^2+28\,a^4\,b\,XX_3^2-8\,b^2\,XX_3^2+40\,a\,b^2\,XX_3^2-76\,a^2\,b^2\,XX_3^2+\\
&64\,a^3\,b^2\,XX_3^2-20\,a^4\,b^2\,XX_3^2+8\,b^3\,XX_3^2-32\,a\,b^3\,XX_3^2+44\,a^2\,b^3\,XX_3^2-20\,a^3\,b^3\,XX_3^2-\\
&8\,a^4\,b\,t^3\,XX_3^2-32\,a^3\,b^2\,t^3\,XX_3^2+48\,a^4\,b^2\,t^3\,XX_3^2-8\,a^2\,b^3\,t^3\,XX_3^2+48\,a^3\,b^3\,t^3\,XX_3^2-\\
&48\,a^4\,b^3\,t^3\,XX_3^2-16\,a^2\,b\,t\,XX_3^{5/2}+40\,a^3\,b\,t\,XX_3^{5/2}-24\,a^4\,b\,t\,XX_3^{5/2}-16\,a\,b^2\,t\,XX_3^{5/2}+\\
&56\,a^2\,b^2\,t\,XX_3^{5/2}-64\,a^3\,b^2\,t\,XX_3^{5/2}+24\,a^4\,b^2\,t\,XX_3^{5/2}+16\,a\,b^3\,t\,XX_3^{5/2}-40\,a^2\,b^3\,t\,XX_3^{5/2}+\\
&24\,a^3\,b^3\,t\,XX_3^{5/2}-16\,a^4\,b^2\,t^4\,XX_3^{5/2}-16\,a^3\,b^3\,t^4\,XX_3^{5/2}+32\,a^4\,b^3\,t^4\,XX_3^{5/2}+\\
&8\,a^3\,b^2\,t^3\,XX_1\,XX_3^{5/2}-8\,a^4\,b^2\,t^3\,XX_1\,XX_3^{5/2}-8\,a^3\,b^3\,t^3\,XX_1\,XX_3^{5/2}+8\,a^4\,b^3\,t^3\,XX_1\,XX_3^{5/2}+\\
&8\,a^3\,b^2\,t^3\,XX_2\,XX_3^{5/2}-8\,a^4\,b^2\,t^3\,XX_2\,XX_3^{5/2}-8\,a^3\,b^3\,t^3\,XX_2\,XX_3^{5/2}+8\,a^4\,b^3\,t^3\,XX_2\,XX_3^{5/2}\big)\end{aligned}$$

Out[•]=
$$2\,(-1+a)\,a^2\,(-1+b)\,(1+b)\,c\,t^3\,\left(a+b-2\,a\,b+2\,a\,b\,t\,\sqrt{XX_3}\right)$$

Out[•]=
$$-2\,(-1+a)\,a^2\,(-1+b)\,b\,c\,t^4\,\left(a+b-2\,a\,b+2\,a\,b\,t\,\sqrt{XX_3}\right)$$

In[•]:=
```
(* check it *)
ζ0s_{1,0}[9] Q_{1,0} + ζ0s_{2,0}[9] Q_{2,0} + ζ0s_{3,0}[9] Q_{3,0} + ζ0s_{4,0}[9] Q_{4,0} + ζ0s[9] +
   ζ0s_{0,0}[9] Q[0, 0] /. {Q_{1,0} → QQcxy[9, 1, 0], Q_{2,0} → QQcxy[9, 2, 0],
   Q_{3,0} → QQcxy[9, 3, 0], Q_{4,0} → QQcxy[9, 4, 0], Q[0, 0] → QQcxy[9, 0, 0]};
% // Simplificate;
ApplyToSeries[Factor, %]
```

Out[•]= $O[t]^{11}$

In[•]:=
```
(* now combine this with the other
 equations to see if anything new is achieved *)
Clear[x0coeffs]
x0coeffs[n_] := x0coeffs[n] = {ξ0s0,0[n], ξ0s1,0[n], ξ0s2,0[n], ξ0s3,0[n], ξ0s4,0[n]}
{xposcoeffs[12] /. x → xs1[12], xposcoeffs[12] /. x → xs3[12],
  xposcoeffs[12] /. x → xs5[12], xnegcoeffs[12] /. x → xs7[12],
  xnegcoeffs[12] /. x → xs9[12], x0coeffs[12]};
Simplificate /@ # & /@ %;
(* there are 6 possible combinations *)
{Det[Drop[%, {1}]], Det[Drop[%, {2}]], Det[Drop[%, {3}]],
 Det[Drop[%, {4}]], Det[Drop[%, {5}]], Det[Drop[%, {6}]]}
```

Out[•]= $\{O[t]^{36}, O[t]^{36}, O[t]^{36}, O[t]^{36}, O[t]^{36}, O[t]^{40}\}$

Section 5.4

In[•]:=
```
(* at this point we can "cheat" and use the fact that we already
 know the solution to Q[0,0] (and know that it is algebraic)
 because it's the same for Kreweras and reverse Kreweras *)
(* so in fact we don't have 5 unknowns, we only have 4 *)
```

In[•]:=
```
(* from our 6 equations we then have 15 possible combinations *)
(* note that all these matrices contain only algebraic terms *)
Clear[all6eqns]
all6eqns[n_] :=
  all6eqns[n] = Simplificate /@ # & /@ (Drop[#, {1}] & /@ {xposcoeffs[n] /. x → xs1[n],
      xposcoeffs[n] /. x → xs3[n], xposcoeffs[n] /. x → xs5[n],
      xnegcoeffs[n] /. x → xs7[n], xnegcoeffs[n] /. x → xs9[n], x0coeffs[n]});
ApplyToSeries[Factor, Det[{all6eqns[4][[1]], all6eqns[4][[2]],
   all6eqns[4][[3]], all6eqns[4][[4]]}]]
ApplyToSeries[Factor, Det[{all6eqns[10][[1]], all6eqns[10][[2]],
   all6eqns[10][[3]], all6eqns[10][[5]]}]]
ApplyToSeries[Factor, Det[{all6eqns[11][[1]], all6eqns[11][[2]],
   all6eqns[11][[3]], all6eqns[11][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[4][[1]], all6eqns[4][[2]],
   all6eqns[4][[4]], all6eqns[4][[5]]}]]
ApplyToSeries[Factor, Det[{all6eqns[4][[1]], all6eqns[4][[2]],
   all6eqns[4][[4]], all6eqns[4][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[9][[1]], all6eqns[9][[2]],
   all6eqns[9][[5]], all6eqns[9][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[4][[1]], all6eqns[4][[3]],
   all6eqns[4][[4]], all6eqns[4][[5]]}]]
ApplyToSeries[Factor, Det[{all6eqns[13][[1]], all6eqns[13][[3]],
   all6eqns[13][[4]], all6eqns[13][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[13][[1]], all6eqns[13][[3]],
   all6eqns[13][[5]], all6eqns[13][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[4][[1]], all6eqns[4][[4]],
   all6eqns[4][[5]], all6eqns[4][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[9][[2]], all6eqns[9][[3]],
   all6eqns[9][[4]], all6eqns[9][[5]]}]]
ApplyToSeries[Factor, Det[{all6eqns[13][[2]], all6eqns[13][[3]],
   all6eqns[13][[4]], all6eqns[13][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[13][[2]], all6eqns[13][[3]],
   all6eqns[13][[5]], all6eqns[13][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[9][[2]], all6eqns[9][[4]],
   all6eqns[9][[5]], all6eqns[9][[6]]}]]
ApplyToSeries[Factor, Det[{all6eqns[13][[3]], all6eqns[13][[4]],
   all6eqns[13][[5]], all6eqns[13][[6]]}]]
```

Out[•]=
$$\frac{16\,(-1+a)^3\,a^{12}\,(a-b)^3\,(-1+b)^5\,b^5\,(-2\,a+a^2+b)\,(-a-b+a\,b)\,(-a-b+2\,a\,b)^5\,c^4\,t^{26}}{(-2+a+b)^4}+O[t]^{28}$$

Out[•]=
$$-\frac{512\,\left((-1+a)^5\,a^{10}\,(a-b)^3\,(-1+b)^4\,b^8\,(-a-b+a\,b)\,(-a-b+2\,a\,b)^5\,c^4\right)\,t^{32}}{(-2+a+b)^4}+O[t]^{34}$$

Out[∘]= $-\frac{1}{(-2+a+b)^4} 128\left((-1+a)^4 a^8 (a-b)^3 (-1+b)^4 b^5 (-a-b+2ab)^5 (56a^2-84a^3+28a^4+112ab-392a^2b+395a^3b-111a^4b+56b^2-308ab^2+563a^2b^2-397a^3b^2+82a^4b^2-56b^3+196ab^3-223a^2b^3+82a^3b^3+a^4b^3)c^4\right)t^{31}+O[t]^{32}$

Out[∘]= $-16\left((-1+a)^2 a^{12} (a-b)^3 (-1+b)^2 b^8 (-a-b+ab)(-a-b+2ab)(-a^2+2a^3+2ab-a^2b-3a^3b-b^2+ab^2-a^2b^2+2a^3b^2)c^4\right)t^{26}+O[t]^{28}$

Out[∘]= $16(-1+a)^3 a^{12}(a-b)^3(-1+b)^5 b^5(-2a+a^2+b)(-a-b+ab)^2 c^4 t^{22}+O[t]^{24}$

Out[∘]= $-512\left((-1+a)^5 a^{10}(a-b)^3(-1+b)^4 b^8(-a-b+ab)^2 c^4\right)t^{28}+O[t]^{29}$

Out[∘]= $-\frac{16\left((-1+a)^3 a^{12}(a-b)^3(-1+b)^2 b^6(-2a+a^2+b)(-a-b+ab)(-a-b+2ab)^5 c^4\right)t^{26}}{(-2+a+b)^4}+O[t]^{28}$

Out[∘]= $O[t]^{33}$

Out[∘]= $O[t]^{33}$

Out[∘]= $16(-1+a)^3 a^{12}(a-b)^3(-1+b)^2 b^6(-2a+a^2+b)(-a-b+ab)^2 c^4 t^{22}+O[t]^{24}$

Out[∘]= $-\frac{512\left((-1+a)^2 a^{11}(a-b)^3(-1+b)^4 b^8(-a-b+ab)(-a-b+2ab)^5 c^4\right)t^{32}}{(-2+a+b)^4}+O[t]^{33}$

Out[∘]= $O[t]^{33}$

Out[∘]= $O[t]^{33}$

Out[∘]= $512(-1+a)^2 a^{11}(a-b)^3(-1+b)^4 b^8(-a-b+ab)^2 c^4 t^{28}+O[t]^{29}$

Out[∘]= $O[t]^{33}$

In[∘]:=
```
(* ok, so it looks like at least 10 of the sets will work *)
(* generating the solutions *)
Clear[xposknown, xnegknown, x0known, all6knowns]
xposknown[n_] := xposknown[n] = Simplificate[σs[n] + σs0,0[n] * QQcxy[n, 0, 0]]
xnegknown[n_] := xnegknown[n] = Simplificate[τs[n] + τs0,0[n] * QQcxy[n, 0, 0]]
x0known[n_] := x0known[n] = Simplificate[ξ0s[n] + ξ0s0,0[n] * QQcxy[n, 0, 0]]
all6knowns[n_] := all6knowns[n] =
  Simplificate /@ {xposknown[n] /. x → xs1[n], xposknown[n] /. x → xs3[n],
    xposknown[n] /. x → xs5[n], xnegknown[n] /. x → xs7[n],
    xnegknown[n] /. x → xs9[n], x0known[n]}
```

In[∘]:=
```
(* Mathematica seems to struggle
 with expanding some of the matrix inverses *)
(* so we will just demonstrate that the first set gives the correct result *)
```

In[•]:=
```
Inverse[{all6eqns[13][[1]], all6eqns[13][[2]], all6eqns[13][[3]],
    all6eqns[13][[4]]}].-{all6knowns[13][[1]], all6knowns[13][[2]],
    all6knowns[13][[3]], all6knowns[13][[4]]} // Simplify;
Simplificate /@ %;
ApplyToSeries[Expand, #] & /@ (% // Simplify)
% - {QQcxy[12, 1, 0], QQcxy[12, 2, 0], QQcxy[12, 3, 0], QQcxy[12, 4, 0]}
```

Out[•]=
$$\{a\,t^2 + (2a + 2a^2 + a^3 + a\,b + a^2\,c + a\,b\,c)\,t^5 + (16a + 16a^2 + 11a^3 + 6a^4 + 2a^5 + 8a\,b + 4a^2\,b + a^3\,b + 3a\,b^2 + a\,b^3 + 4a^2\,c + 4a^3\,c + 2a^4\,c + 4a\,b\,c + 5a^2\,b\,c + a^3\,b\,c + 3a\,b^2\,c + a\,b^3\,c + a^3\,c^2 + 2a^2\,b\,c^2 + a\,b^2\,c^2)\,t^8 + O[t]^9,$$
$$(a + a^2)\,t^4 + (8a + 8a^2 + 5a^3 + 2a^4 + 2a\,b + a^2\,b + a^2\,c + a^3\,c + a\,b\,c + a^2\,b\,c)\,t^7 + O[t]^8,$$
$$(2a + 2a^2 + a^3)\,t^6 + O[t]^7,\ O[t]^6\}$$

Out[•]=
$$\{O[t]^9,\ O[t]^8,\ O[t]^7,\ O[t]^6\}$$