Powered Catalan numbers

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The sequence

OEIS sequence A113227:

$$(u_n)_{n\geq 0} = (1, 1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, 8555388, 72442465, \ldots)$$

First (?) observed by David Callan circa 2005 as the number of permutations of length n which avoid the generalised pattern 1-23-4.

$\rightarrow \pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathcal{S}_n$ such that there are no $i, j, k \in \{1, \ldots, n\}$ with

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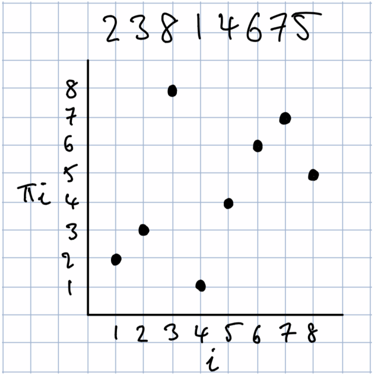

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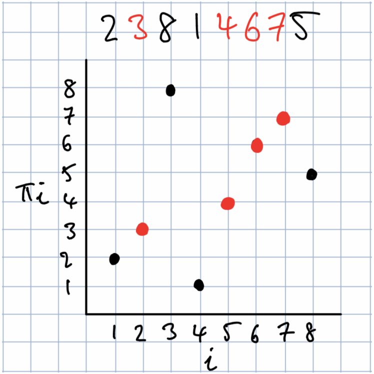

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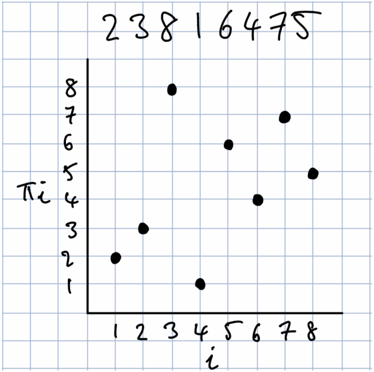


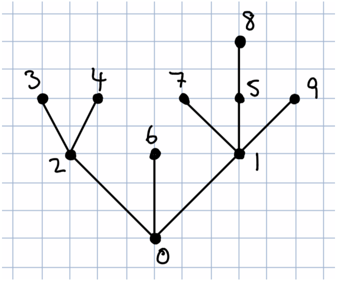
A recurrence for trees

In 2010 Callan found a two-index recurrence for u_n:

$$u_n = \sum_{k=1}^{n} u_{n,k}, \quad \text{where} \quad u_{n,k} = u_{n-1,k-1} + k \sum_{j=k}^{n-1} u_{n-1,j} \quad \text{for} \quad 1 \leq k \leq n$$

with initial conditions $u_{0,0} = 1$ and $u_{n,0} = 0$ for $n \geq 1$.

In the process he used a (very complicated) bijection with increasing ordered trees with increasing leaves:



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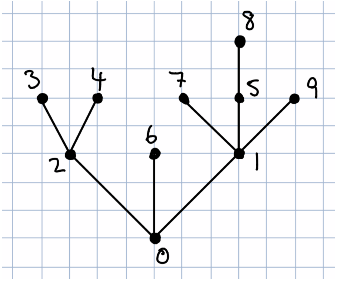

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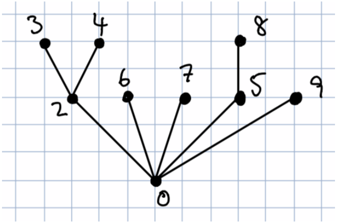

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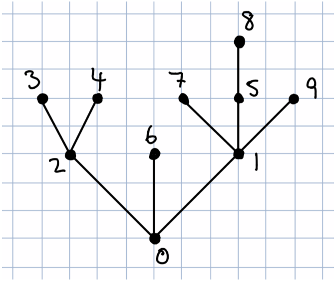

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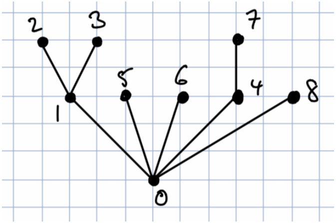

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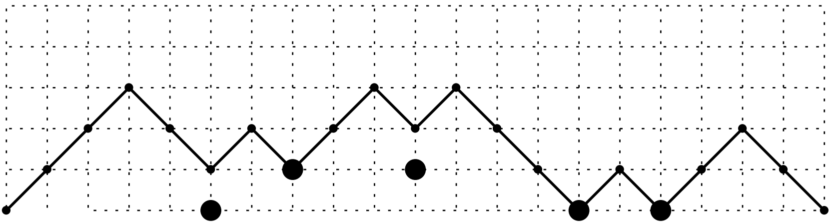
Valley-marked Dyck paths

He also noticed that u_n counts valley-marked Dyck paths (VMDPs) of length $2n$.



Each valley at height h gets a mark at one of $0, 1, \ldots, h$.

$u_{n,k}$ is the number of VMDPs of length $2n$ with final descent k.

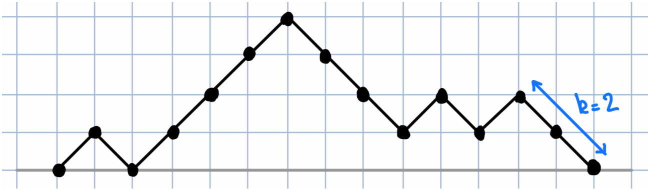
Generating trees

Why the name powered Catalan numbers?

A generating tree or succession rule is method for illustrating the recursive growth of combinatorial objects. For example, one recurrence for the Catalan numbers is

$$C_n = \sum_{k=1}^{n} C_{n,k}, \quad \text{where} \quad C_{n,k} = \sum_{j=k-1}^{n-1} u_{n-1,j} \quad \text{for} \quad 1 \leq k \leq n$$

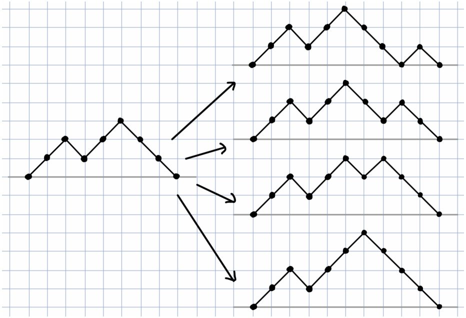
with initial conditions $C_{0,0} = 1$ and $C_{n,0} = 0$ for $n \geq 1$. ($C_{n,k}$ is the number of Dyck paths of length $2n$ with final descent k.)



Generating trees cont'd

Label each Catalan object (eg. Dyck path) with (k). Then the initial object has label (0), and something with label (k) can be grown into something with label $(1), (2), \ldots, (k+1)$. Write this as

$$\Omega_{\text{Cat}} = \begin{cases} (0) \\ (k) \rightsquigarrow (1), (2), \ldots, (k+1) \end{cases}$$



Then Callan's recurrence for $u_{n,k}$ is

$$\Omega_{\text{pCat}} = \begin{cases} (0) \\ (k) \rightsquigarrow (1), (2), (2), (3), (3), (3), \ldots, (k)^k, (k+1) \end{cases}$$

Another succession rule for u_n

We came across this sequence while studying another type of object: inversion sequences. These encode permutations π by their inversions: pairs (i, j) such that $i < j$ and $\pi_i > \pi_j$.

In particular, u_n is the number of 110-pattern-avoiding inversion sequences of length n.

But along the way we found another succession rule which also generates u_n:

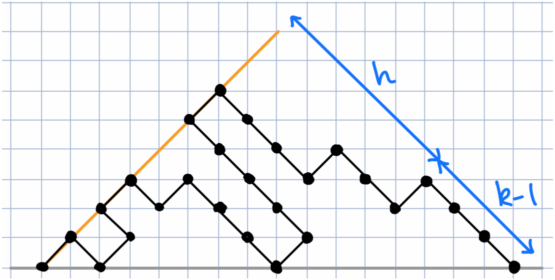
$$\Omega_{\text{steady}} = \begin{cases} (0,1) \\ (h,k) \rightsquigarrow (h+k-1,2),(h+k-2,3),\ldots,(h+1,k), \\ \qquad\qquad (0,k+1),\ldots,(0,h+k+1) \end{cases}$$

And in fact it is must simpler to show that (1-23-4)-pattern-avoiding permutations grow according to Ω_{steady} than Callan's proof that they grow according to Ω_{pCat}.

Steady paths

Why steady? A steady path of size n is a generalised Dyck path which may also step "back" $(-1, 1)$ (but remains self-avoiding), with the constraint that any consecutive pair of steps UU or BU creates a diagonal "wall" that the walk must stay below. (The wall is initially the line $y = x$.)

n is the number of U steps.



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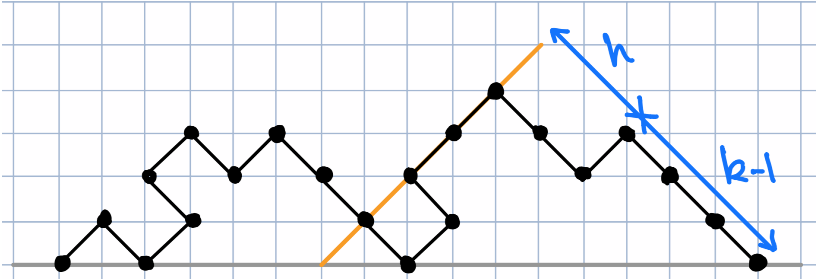
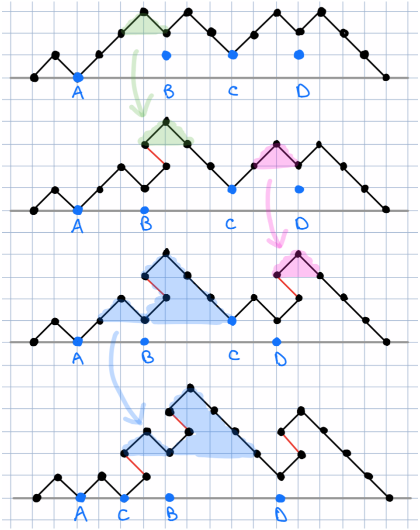


In Ω_{steady}, the final descent is $k - 1$ and h is the gap between the final peak and the wall.

A bijection between Ω_{pCat} and Ω_{steady}

All of the classes counted by u_n (there are others) seem to naturally grow according to either Ω_{pCat} or Ω_{steady}. Is there a simple way to connect the two?

Yes! Just mess around with the paths...



A bijection between Ω_{pCat} and Ω_{steady} cont'd

Multiple statistics are conserved:

- number of U steps along the first diagonal $y = x$
- total height of marks $\mapsto$ total number of B steps
- number of valleys with highest possible mark $\mapsto$ number of valleys at height 0

Conclusion

The sequence (u_n) arose independently (at least) twice, both times counting something algebraic: (1-23-4)-pattern-avoiding permutations and 110-pattern-avoiding inversion sequences.

There are (at least) two succession rules Ω_{pCat} and Ω_{steady}, and different classes seem to grow naturally according to one or the other.

By choosing the right representatives for each (VMDPs for Ω_{pCat}, steady paths for Ω_{steady}), the bijection from one side to the other becomes much clearer!

Conjecture: (23-1-4)-pattern-avoiding permutations are also counted by u_n!

Reference: NRB, Mathilde Bouvel, Veronica Guerrini & Simone Rinaldi, *Enumerating five families of pattern-avoiding inversion sequences; and introducing the powered Catalan numbers*, arXiv:1808.04114.

Inversion sequences

We found the sequence in another place: inversion sequences.

An inversion sequence of length n is a sequence $(e_1, \dots, e_n)$ such that $0 \leq e_i < i$.

The number of inversion sequences $|\mathcal{I}_n|$ of length n is $n!$, and they are in bijection with permutations:

$$\pi = (\pi_1, \dots, \pi_n) \mapsto (t_1, \dots, t_n), \quad \text{where} \quad t_i = |\{j : j > i \text{ and } \pi_j < \pi_i\}$$

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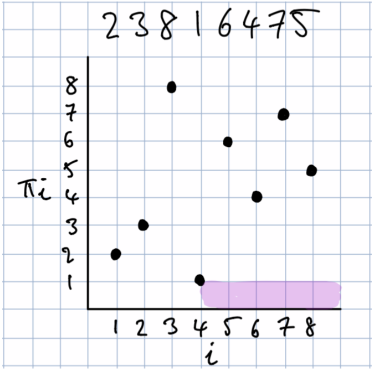

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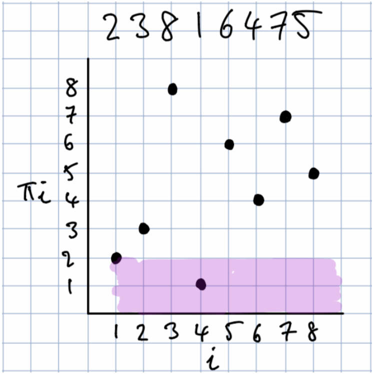

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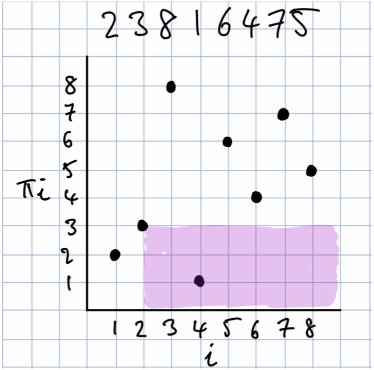

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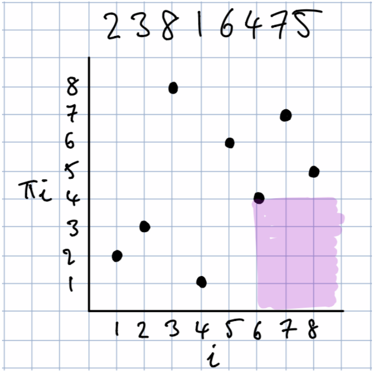

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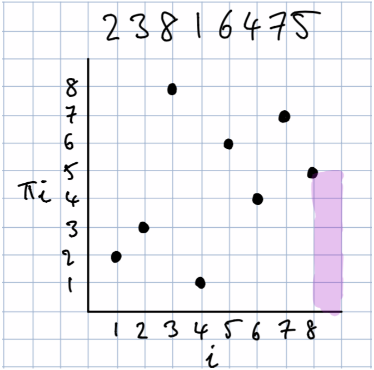

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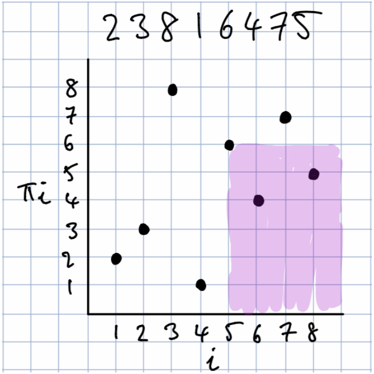

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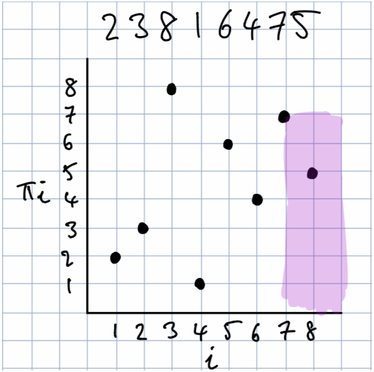

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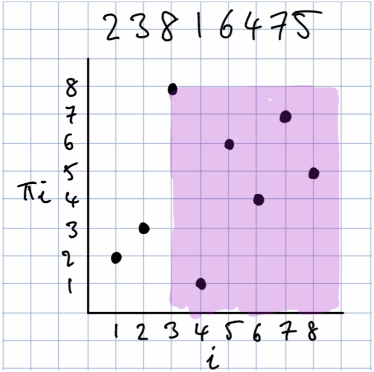

$$t = (0, 1, 1, 0, 0, 2, 1, 5)$$

Pattern-avoiding inversion sequences

Pattern-avoiding inversion sequences were introduced by Mansour & Shattuck in 2015, and later studied by Corteel et al (2016). Duncan & Steingrimsson (2011) also studied patterns in ascent sequences, which are a special case.

The sequence (u_n) counts 110-avoiding inversion sequences: $\rightarrow t = (t_1, \ldots, t_n) \in \mathcal{I}_n$ such that there are no $i, j, k \in \{1, \ldots, n\}$ with

$$i < j < k \quad \text{and} \quad t_i = t_j > t_k.$$

And in fact $u_{n,k}$ is the number of 110-avoiding inversion sequences with k 0's.

The corresponding permutations do not seem to be pattern-avoiding (that we know of).