# Alignment percolation

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Mathematical Physics Seminar
University of Melbourne

# Outline

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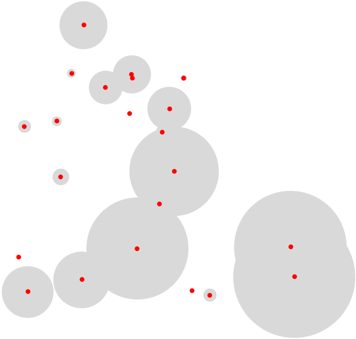

# Introduction I: Some models in continuum percolation

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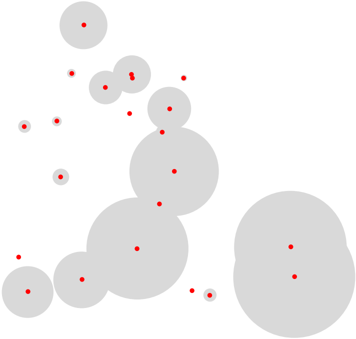

Let

$$C = \bigcup_{x \in X} B_x \qquad \text{and} \qquad V = \mathbb{R}^d \setminus C$$

**Theorem [Menshikov 1986]**

*For $d \geq 2$ and given distribution for the $\rho_i$: there exists a critical intensity $0 < \lambda_c < \infty$ such that $C$ contains an infinite component a.s. for $\lambda > \lambda_c$.*

**Theorem [Roy 1990]**

*For $d = 2$ and given distribution for the $\rho_i$: when $\lambda > \lambda_c$ we have $C$ containing an infinite component while $V$ does not, and when $\lambda < \lambda_c$ we have $V$ containing an infinite component while $C$ does not.*

For $d \geq 3$ it is expected that there is an 'intermediate' region $(\lambda^*, \lambda_c)$ where both $C$ and $V$ contain infinite components.

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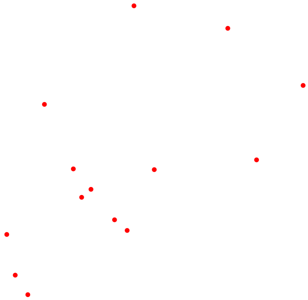

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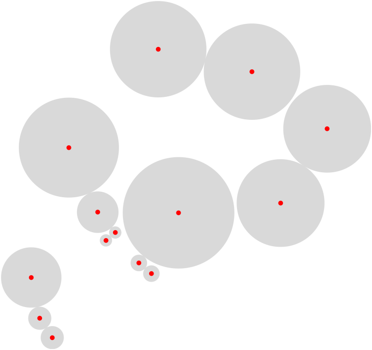

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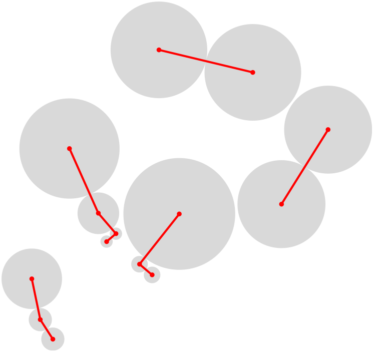

This then defines an infinite random graph $G$, whose vertex set is $X$ and where edge $xy$ is present if $x$ and $y$'s balls touch.

**Theorem [Häggström & Meester 1997]**

*The graph $G$ a.s. has no infinite component.*

(Note that by scale invariance the intensity $\lambda$ and the rate of growth are irrelevant.)

**Lemma [Häggström & Meester 1997]**

*The graph $G$ is a.s. a forest.*

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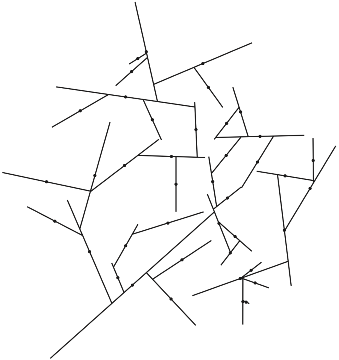

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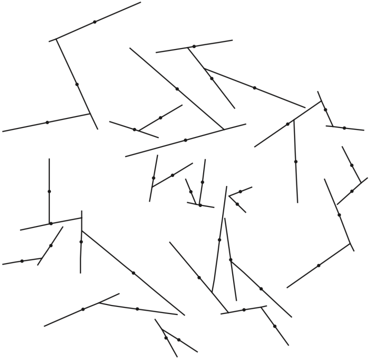
- Model 1: Stop growing $L_x$ when it collides with another line (at either end).
- Model 2: Stop growing $L_x$ when it collides with another line **or** another line collides with $L_x$.



(Image from [Daley et al. 2016])

This defines an infinite random graph $G$, with vertex set $X$ and where vertices $x, y$ are joined by an edge if their line segments touch.

**Theorem** [Daley et al. 2016]

*For Model 2, the graph $G$ a.s. has no infinite component.*

**Conjecture** [Daley et al. 2016]

*For Model 1, the graph $G$ a.s. has no infinite component.*

Note that again the intensity $\lambda$ and rate of growth of the line segments are irrelevant.

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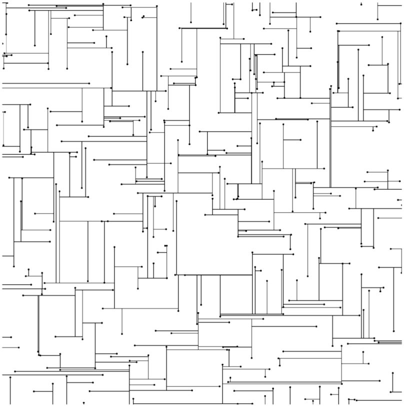

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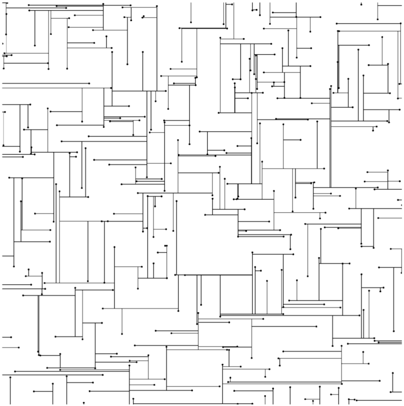

(Figure from [Hirsch 2016])

This defines an infinite random graph $G$ in the same manner as the previous models.

**Theorem** [Hirsch 2016]

*The graph $G$ a.s. has no infinite component.*

**Question:** Is there a lattice version of these lilypond models, ie. with points in $\mathbb{Z}^d$ instead of $\mathbb{R}^d$? Does such a system ever percolate?

# Introduction II: A quick percolation refresher

(Henceforth always assume $d \geq 2$.)

**Bond percolation:** Let $E$ be the set of adjacent pairs of vertices in $\mathbb{Z}^d$. Define $e \in E$ to be open with probability $p$ and closed with probability $1 - p$. Let $Y \subset E$ be the set of open pairs, and let $X$ be the set of points which are in an open pair. Define graph $G$ to have vertex set $X$ and edge set $Y$.

Let $\theta(p) = \mathbb{P}(\text{origin is part of an infinite open cluster of } G)$.

**Theorem [Broadbent & Hammersley 1957; Harris 1960]**

*Define $p_c^{\text{bond}}$ to be*

$$p_c^{\text{bond}} = \sup\{p : \theta(p) = 0\}.$$

*Then $0 < p_c^{\text{bond}} < 1$. $G$ a.s. has an infinite component if $p > p_c^{\text{bond}}$, and $G$ a.s. has no infinite component if $p < p_c^{\text{bond}}$. When it exists, the infinite component is a.s. unique.*

For $d = 2$ we have $p_c^{\text{bond}} = \frac{1}{2}$ [Kesten 1980]

**Site percolation:** Each $x \in \mathbb{Z}^d$ is open with probability $p$ and closed with probability $1 - p$. Let $X$ be the set of open vertices. Define graph $G$ to have vertex set $X$, with vertices $x, y \in X$ joined by an edge if $|x - y| = 1$.

**Theorem**

*With $\theta$ defined as above, define $p_c^{\text{site}}$ to be*

$$p_c^{\text{site}} = \sup\{p : \theta(p) = 0\}.$$

*Then $0 < p_c^{\text{site}} < 1$. $G$ a.s. has an infinite component if $p > p_c^{\text{site}}$, and $G$ a.s. has no infinite component if $p < p_c^{\text{site}}$. When it exists, the infinite component is a.s. unique.*

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**Mixed percolation:** Vertex set $X$ is the same as in site percolation (with parameter $p$). But now if $x, y \in X$ are adjacent in $\mathbb{Z}^d$, add the edge $xy$ with probability $\lambda$.

Let $\theta(\lambda, p) = \mathbb{P}(\text{origin is part of an infinite open cluster of } G)$.

**Theorem [Chayes & Schonmann 2000]**

*Define $\lambda_c^{\text{mixed}}(p)$ to be*

$$\lambda_c^{\text{mixed}}(p) = \sup\{\lambda : \theta(\lambda, p) = 0\}.$$

*Then $\lambda_c^{\text{mixed}}(p) = 1$ for $0 \le p \le p_c^{\text{site}}$ and $\lambda_c^{\text{mixed}}(1) = p_c^{\text{bond}}$. On the interval $[p_c^{\text{site}}, 1]$, $\lambda_c^{\text{mixed}}$ is Lipschitz continuous and strictly decreasing.*

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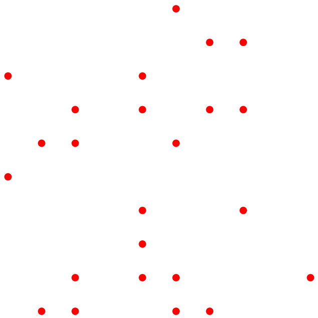

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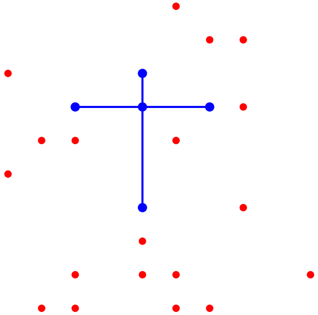

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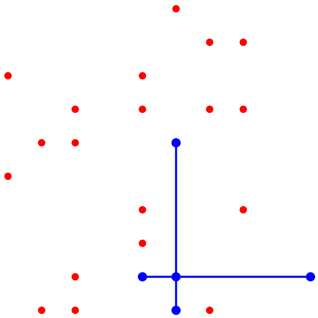

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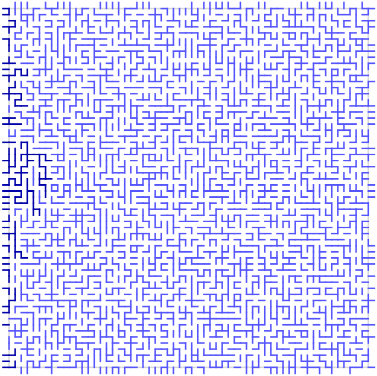

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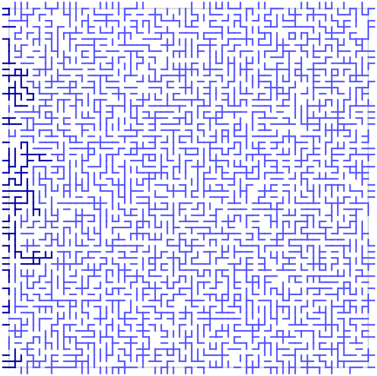

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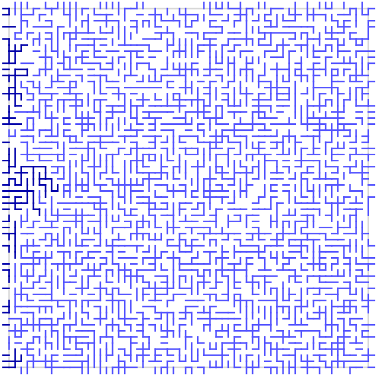

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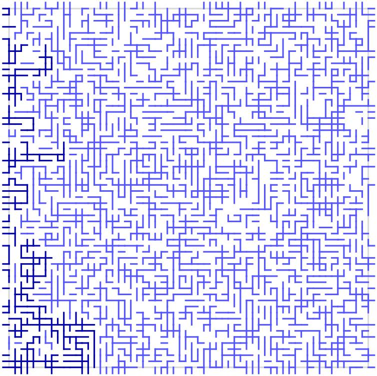

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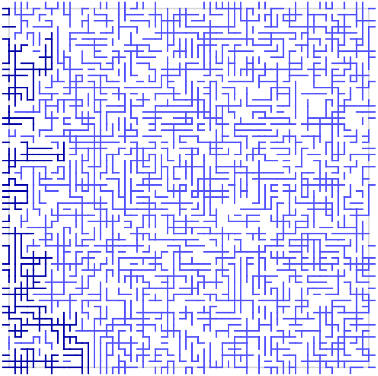

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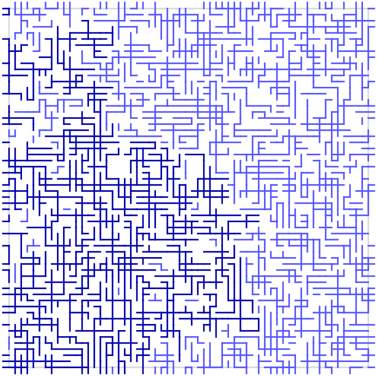

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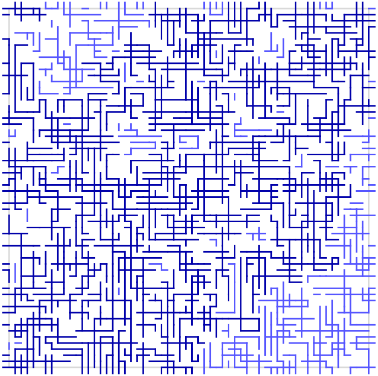

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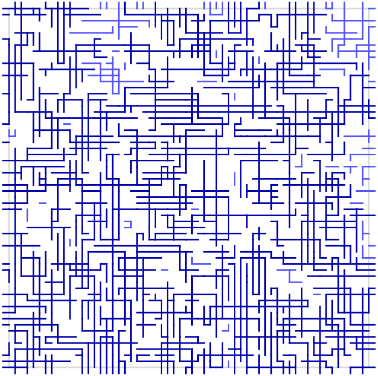

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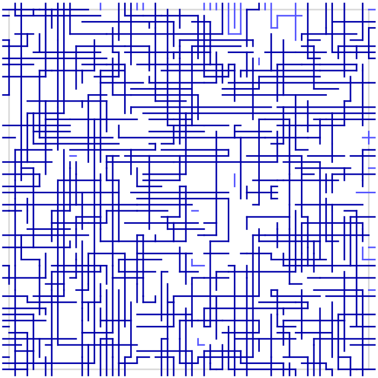

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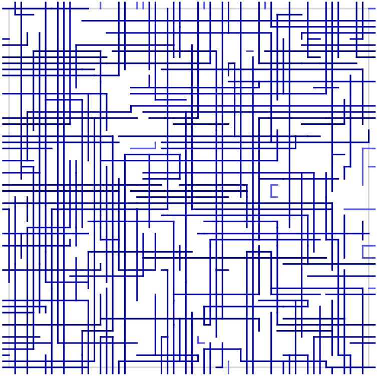

$p = 0.1$

# The one-choice model

With $\omega$ a realisation of site percolation with parameter $p$ and $F(\omega)$ the set of feasible pairs, we take a probability distribution on $F(\omega)$ as follows:

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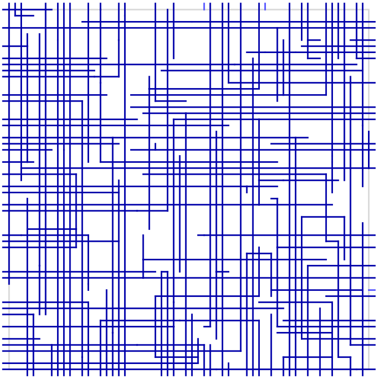

$p = 0.05$

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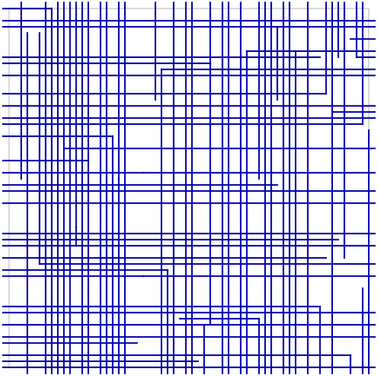

$p = 0.01$

# Some easy facts

Any edge of $\mathbb{Z}^d$ is blue with probability

$$\lambda := 1 - \left(1 - \frac{1}{2d}\right)^2.$$

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The probability that edges $(o, e_1)$ and $(o, -e_1)$ are blue is

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and for edges $(o, e_1)$ and $(0, e_2)$ it is

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Note that restricting the $d$-dimensional model to the $(d-k)$-dimensional sublattice $\{0\}^k \times \mathbb{Z}^{d-k}$ does not result in the one-choice model on $\mathbb{Z}^{d-k}$.

# Main results

**Theorem 1**

*For the one-choice model, there exist* $0 < p_0^{\text{o-c}}(d) \leq p_1^{\text{o-c}}(d) < 1$ *such that*

**(i)** *if* $p \in (0, p_0^{\text{o-c}}(d))$ *there exists a.s. a unique infinite blue cluster;*

**(ii)** *if* $p \in (p_1^{\text{o-c}}(d), 1]$ *there exists a.s. no infinite blue cluster.*

**Conjecture 1**

*For the one-choice model,*

**(i)** *there exists* $p_c^{\text{o-c}}(d) \in (0,1)$ *such that for* $p \in (0, p_c^{\text{o-c}}(d))$ *there exists a.s. a unique infinite blue cluster, and for* $p \in (p_c^{\text{o-c}}(d), 1]$ *there exists a.s. no infinite blue cluster;*

**(ii)** $p_c^{\text{o-c}}(d)$ *is strictly increasing with* $d$*;*

**(iii)** *the probability* $\theta(p)$ *that the origin lies in an infinite blue cluster is non-increasing in* $p$.

Our best numerical estimates are $p_c^{\text{o-c}}(2) \approx 0.505$ and $p_c^{\text{o-c}}(3) \approx 0.862$.

Theorem 1 (ii) follows by coupling with a **corrupted compass model** [Hirsch et al. 2018].

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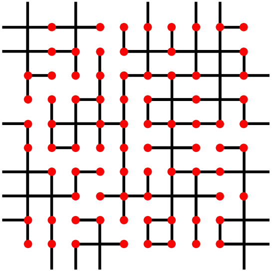
This is a version of bond percolation, with edges made open as follows: for each site $x \in \mathbb{Z}^d$,

- with probability $p$, choose one of the $2d$ edges incident on $x$ and declare it open
- with probability $1 - p$, declare all of the $2d$ edges incident on $x$ to be open

(an edge may be declared open twice)



$p = 0.9$

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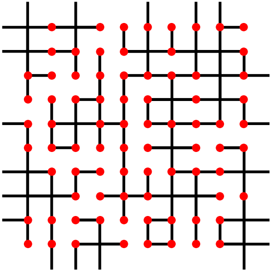

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**Theorem [Hirsch et al. 2018]**

*For the corrupted compass model, there exists a $p_c^{comp}(d) \in (0,1)$ such that the set of open edges a.s. does not percolate for $p \in (p_c^{comp}(d), 1]$.*

The one-choice model can then be coupled with the corrupted compass model: the underlying site percolation sites $\omega$ generate one open edge in the direction of their blue segment, and all other sites generate $2d$ open edges.

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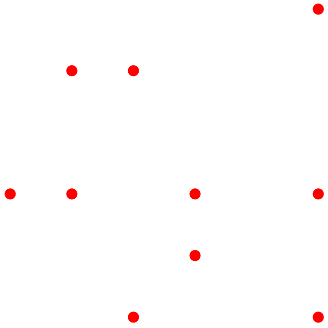

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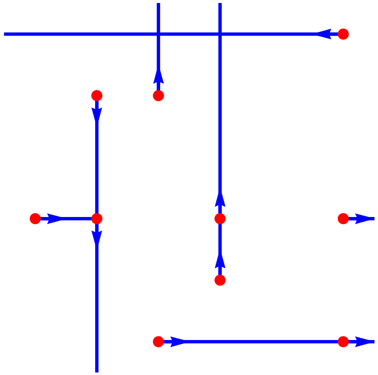

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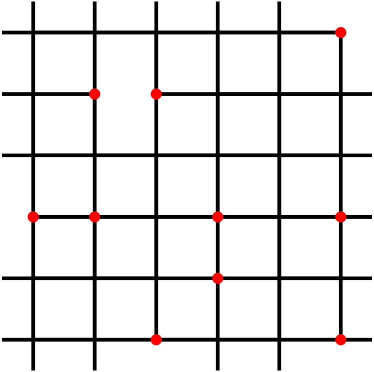

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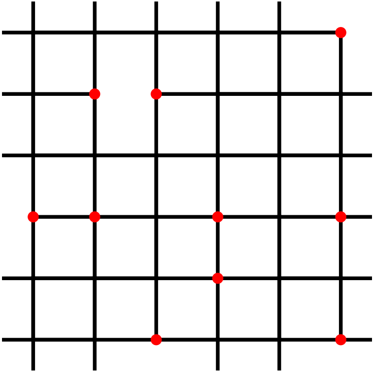

With $B$ the set of blue edges and $T$ the open edges, we have $B \subseteq T$, and hence $B$ a.s. does not percolate if $p \in (p_c^{\text{comp}}(d), 1]$.

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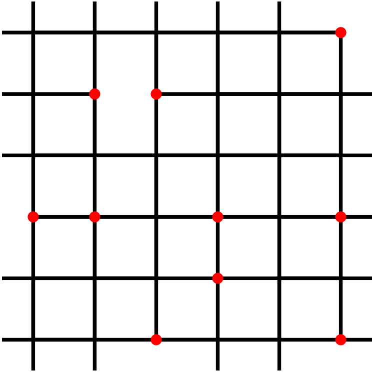

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Will return to Theorem 1 (i) later...

# Outline

# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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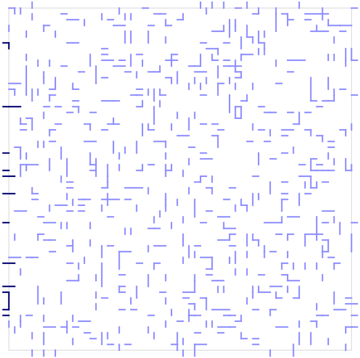

$(p, \lambda) = (0.8, 0.1)$

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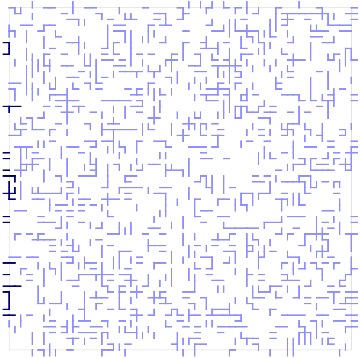

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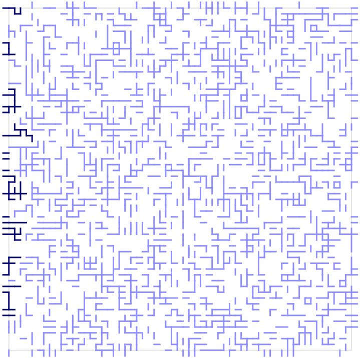

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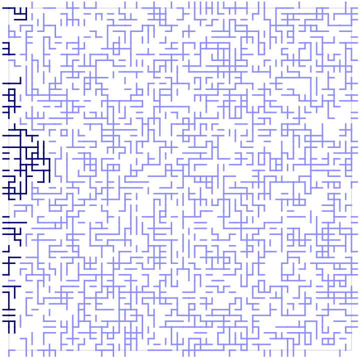

$(p, \lambda) = (0.8, 0.4)$

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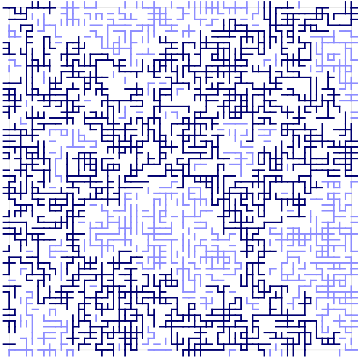

$(p, \lambda) = (0.8, 0.45)$

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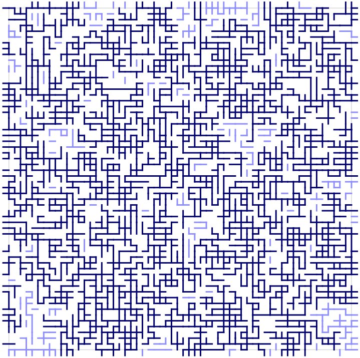

$(p, \lambda) = (0.8, 0.5)$

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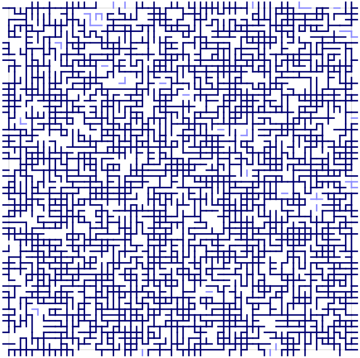

$(p, \lambda) = (0.8, 0.6)$

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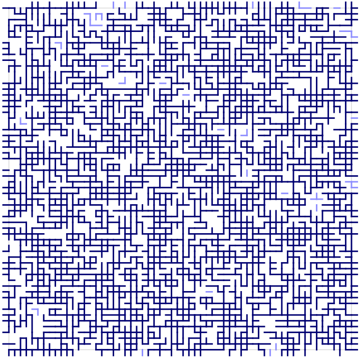
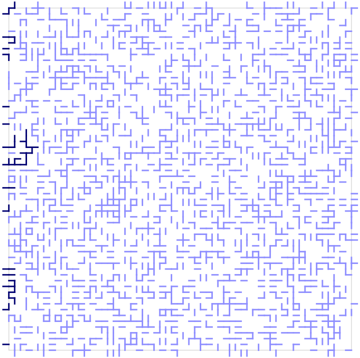


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (1, 0.3)$

# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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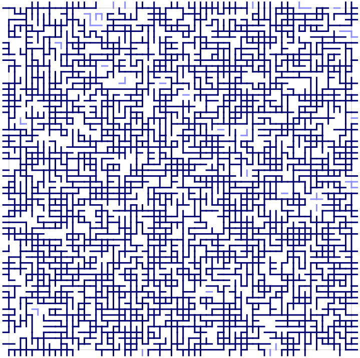
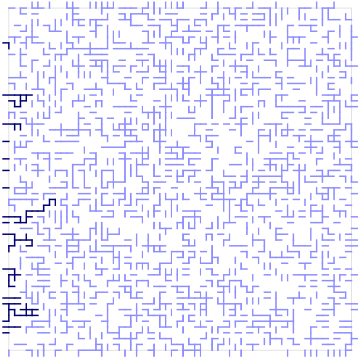


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.9, 0.3)$

# The independent model

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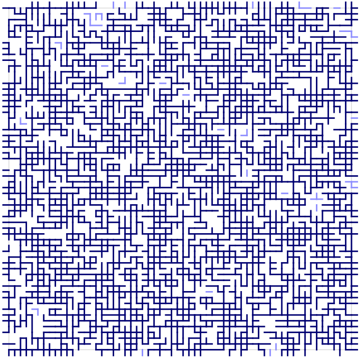
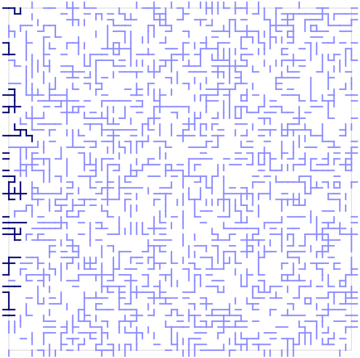


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.8, 0.3)$

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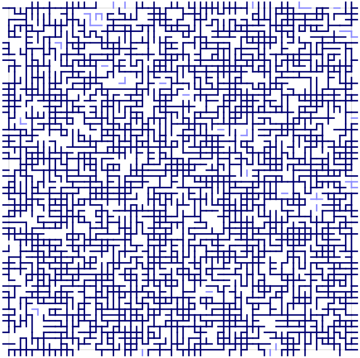
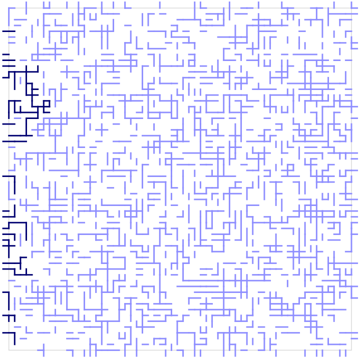


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.7, 0.3)$

# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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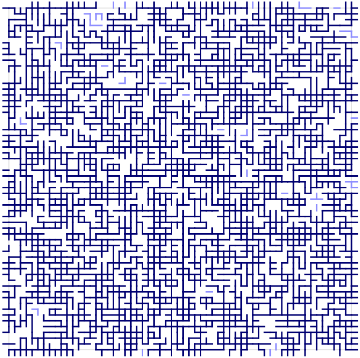
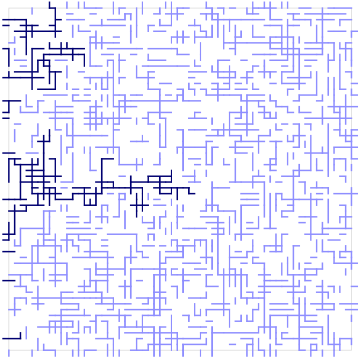
$S(\omega)$ is the set of blue segments. Sites and edges of $\mathbb{Z}^d$ are declared blue as before.



$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.6, 0.3)$

# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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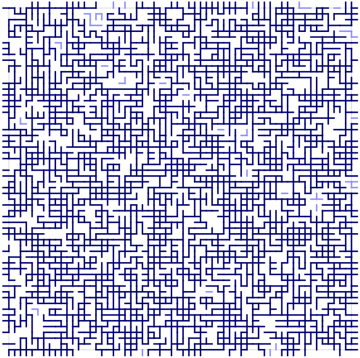
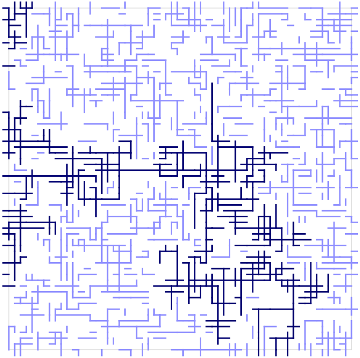


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.5, 0.3)$

# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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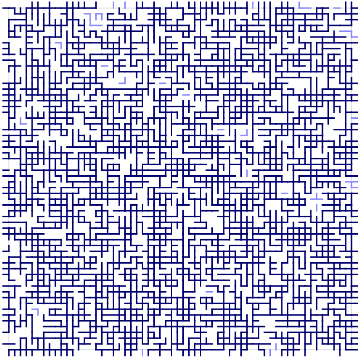
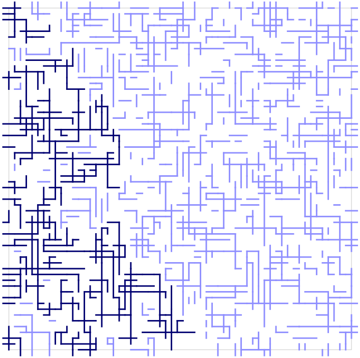


$(p, \lambda) = (0.8, 0.6)$



$(p, \lambda) = (0.45, 0.3)$

# The independent model

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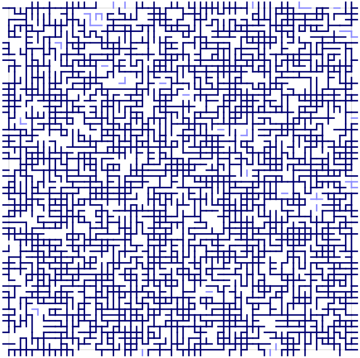

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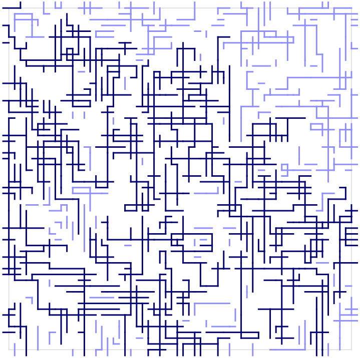

$(p, \lambda) = (0.4, 0.3)$

# The independent model

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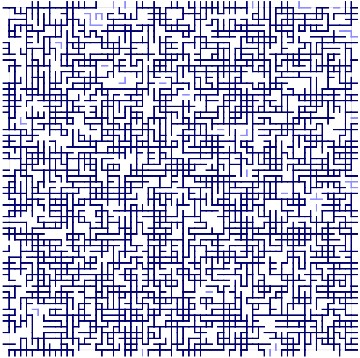

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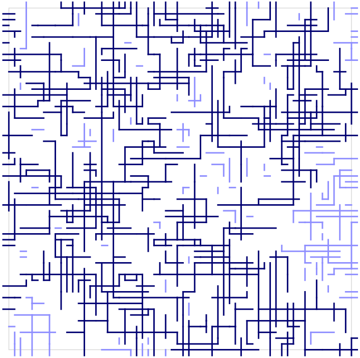

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# The independent model

Again start with $\omega$ a realisation of site percolation with parameter $p$, and $F(\omega)$ the set of feasible pairs. We take a different distribution on $F(\omega)$:

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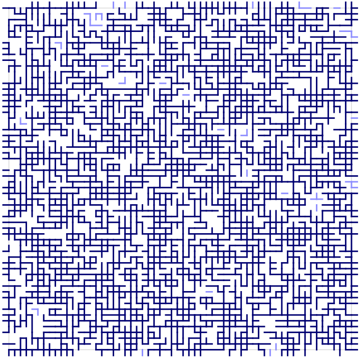
$S(\omega)$ is the set of blue segments. Sites and edges of $\mathbb{Z}^d$ are declared blue as before.



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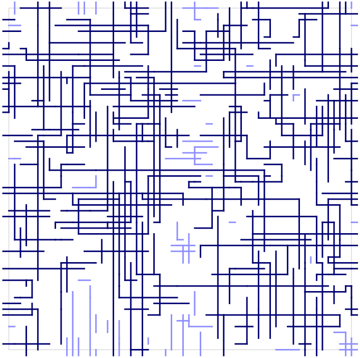

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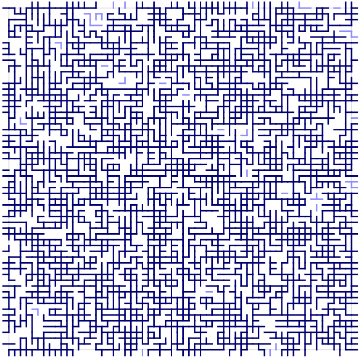

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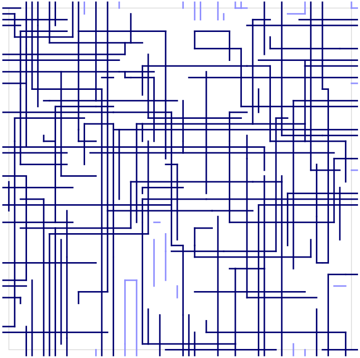

$(p, \lambda) = (0.1, 0.3)$

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For two distinct edges $e$ and $e'$,

$$\mathrm{corr}\left(\mathbf{1}_{\{e \text{ is blue}\}}, \mathbf{1}_{\{e' \text{ is blue}\}}\right) = \begin{cases} (1-p)^k & e \text{ and } e' \text{ are collinear} \\ 0 & \text{otherwise} \end{cases}$$

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Note that restricting the $d$-dimensional model to the $(d-k)$-dimensional sublattice $\{0\}^k \times \mathbb{Z}^{d-k}$ does result in the independent model on $\mathbb{Z}^{d-k}$.

# Main results

**Theorem 2**

*For the independent model with parameters* $(p, \lambda)$

**(i)** *if* $\lambda < p/(2d-1)$ *there is a.s. no infinite blue cluster;*

**(ii)** *there exists an absolute constant* $c > 0$ *such that, if* $\lambda > c \log_2(1/(1-p))$, *there exists a.s. a unique infinite blue cluster;*

**(iii)** *for* $p > p_c^{\mathrm{site}}$ *and* $\lambda > \lambda_c^{\mathrm{mixed}}(p)$, *there exists a.s. a unique infinite blue cluster.*

**Conjecture 2**

*For the independent model with parameters* $(p, \lambda)$, *there exists* $\lambda_c^{\mathrm{ind}}(p, d) \in (0, 1)$ *such that*

**(i)** $\lambda_c^{\mathrm{ind}}(\cdot, d)$ *is continuous and strictly increasing on* $(0, 1]$*;*

**(ii)** *for* $p > 0$ *and* $\lambda < \lambda_c^{\mathrm{ind}}(p, d)$, *there exists a.s. no infinite blue cluster;*

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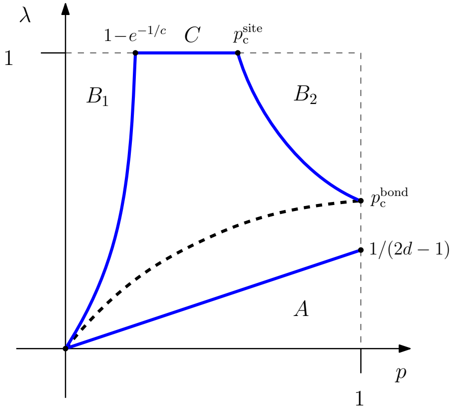
# Phase diagram




- Region $A$: no percolation (Theorem 2 (i))
- Regions $B_1$ and $B_2$: percolates (Theorem 2 (ii) and (iii))
- Line $C$: percolates (trivial)
- Dashed line: conjectured $\lambda_c^{\text{ind}}(p, d)$

Note that numerical calculations for $c$ are very big, so region $B_1$ is actually very small.

Theorem 2 (iii) is straightforward: if we only consider blue segments of length 1, we get mixed percolation with parameters $(p, \lambda)$, which percolates a.s. for $\lambda > \lambda_c^{\text{mixed}}(p)$. Including longer segments does not decrease the probability of percolation.

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Total size of the resulting cluster is (stochastically) bounded above by a two-type branching process, with expected numbers of children $\mu_1$ and $\mu_2$. This process dies out a.s. if $\mu_1, \mu_2 < 1$, ie. if $\lambda < p/(2d - 1)$.

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*Let $\mathbb{E}$ be the edges of $\mathbb{Z}^d$, and let $\mu$ be a translation-invariant probability measure on $\{0,1\}^{\mathbb{E}}$. Suppose*

$$0 < \mu(e \text{ is blue} \,|\, \mathcal{T}_e) < 1 \quad \mu\text{-a.s. for all } e \in \mathbb{E}$$

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The one-choice model does not satisfy the f.e.p., and proving uniqueness is complicated...

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We show that for sufficiently small $p$, the set of blue edges arising from $\mu$ satisfying C1 and C2 percolates a.s. in $\mathbb{Z}^2$.

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A site percolation process is 1-dependent if the state of a site depends on its immediate neighbours. It is known [Ligett et al. 1997] that there exists $\rho \in (0, 1)$ such that a process percolates a.s. for density $> \rho$.

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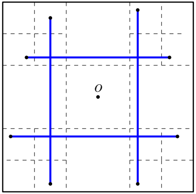

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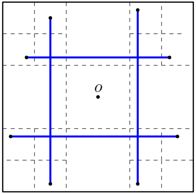


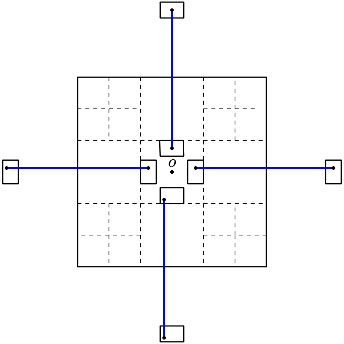

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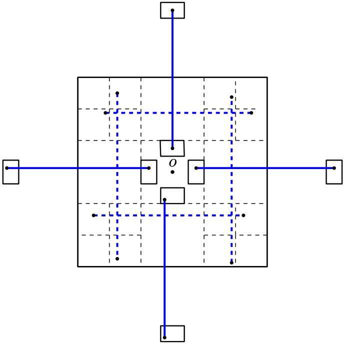

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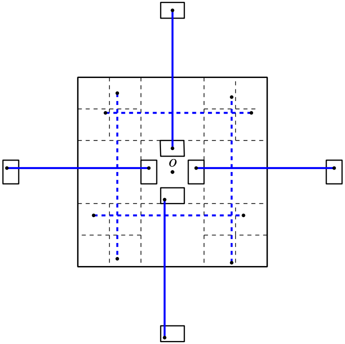


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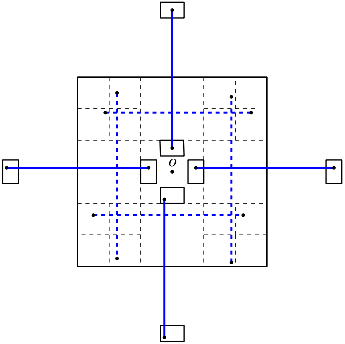


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Choose $c > 0$ so that $1 - 4e^{-c_1 \lambda r} - 4e^{-c_2 \lambda r} > \rho$ when $\lambda r > c$

$$\Rightarrow \lambda > c \log_2(1/(1-p)).$$

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**arXiv:1908.07203**

Thank you for listening!

# Uniqueness of infinite cluster for one-choice model

Let $N$ be the number of infinite clusters. Three main steps:

**A.** Show that $N$ is a.s. constant: follows from the fact that the probability measure is ergodic, and $N$ is translation-invariant.

**B.** Show that $\mathbb{P}(N \in \{0, 1, \infty\}) = 1$: if $N \geq 2$, find a finite box intersected by two infinite clusters, and perform surgery to join them together.

**C.** Show that $\mathbb{P}(N = \infty) = 0$: if $N \geq 3$, find a finite box intersected by three infinite clusters, and perform surgery to join them together. This implies the existence of a **trifurcation**, but there is a theorem by [Burton & Keane 1989] proving that this is impossible.