Solvable self-avoiding walk and polygon models with large growth rates

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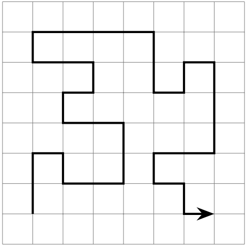
Introduction

Let c_n be the number of self-avoiding walks of length n on the square lattice $\mathbb{Z}^2$.



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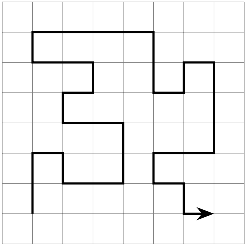

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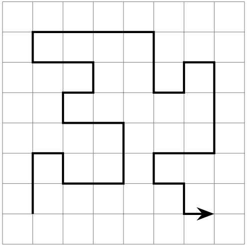

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A very hard problem!

Asymptotics

We still know some things about c_n. Because any SAW of length $m + n$ can be split into two smaller ones,

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Theorem (Hammersley 1957)

The limit

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That is, the rate of growth of c_n is exponential:

Corollary

$$c_n = e^{o(n)}\mu^n.$$

Asymptotics

The constant μ is called the growth constant (sometimes connective constant), and depends on the lattice in question. On the square lattice,

$$\mu \approx 2.63815853031.$$

It is not known exactly for any regular lattice in ≥ 2 dimensions, except the honeycomb (hexagonal) lattice, where $\mu = \sqrt{2+\sqrt{2}}$ (Duminil-Copin & Smirnov 2012).

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In low dimension the subexponential term is not known exactly, but it is widely expected to follow a power law.

Conjecture

In 2 dimensions,

$$c_n \sim An^{\gamma-1}\mu^n$$

The constant A is the amplitude and depends on the lattice, while the exponent γ should only depend on the dimension. In $d = 2$ there is good reason to expect $\gamma = 43/32$, while for $d \geq 5$ it is known that $\gamma = 1$.

Behaviour of the generating function

Define the generating function

$$C(z) = \sum_{n \geq 0} c_n z^n.$$

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Then $z_c = \mu^{-1}$ is the radius of convergence of $C(z)$, and (in 2d) we should have

$$C(z) \sim A'(1 - \mu z)^{-43/32}$$

for $z \sim z_c$ and some constant A'.

Solvable subclasses

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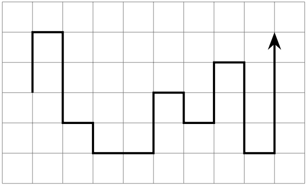
Instead, can look for subclasses of SAWs which are solvable. They may

- shed light on the overall SAW problem
- lead to physical models for which more precise information can be obtained
- lead to new techniques for enumeration, analysis, etc.
- be interesting in their own right!

A simple example: partially directed walks

The simplest classes are obtained by forbidding one or more step directions. eg. a NES-partially directed walk can step $\uparrow, \rightarrow$ and $\downarrow$ but not $\leftarrow$.



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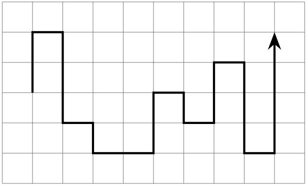

Easy to construct these recursively:

- either a walk has no $\rightarrow$ steps, and so is empty or just $\uparrow$ steps or just $\downarrow$ steps; or
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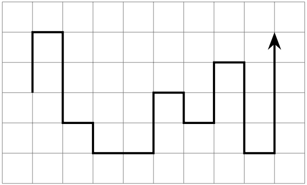

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This can be written as an equation involving the generating function $P(z)$:

$$P(z) = 1 + \frac{2z}{1-z} + z\left(1 + \frac{2z}{1-z}\right)P(z).$$

A simple example: partially directed walks

So

$$P(z) = \frac{1+z}{1-2z-z^2}$$
$$= 1 + 3z + 7z^2 + 17z^3 + 41z^4 + 99z^5 + 239z^6 + \dots$$

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and the number c_n^{PDW} of PDWs of length n is

$$c_n^{\mathrm{PDW}} = \frac{\left(-2+\sqrt{2}\right)\left(1-\sqrt{2}\right)^n + \left(2+\sqrt{2}\right)\left(1+\sqrt{2}\right)^n}{2\sqrt{2}}$$
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Question: How close can we get to $\mu \approx 2.63815853031$ with a solvable model?

Prudent walks

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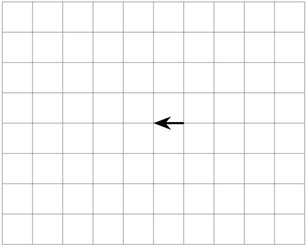

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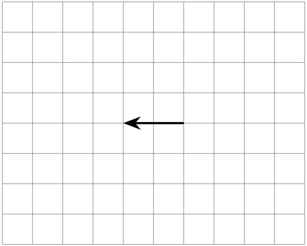

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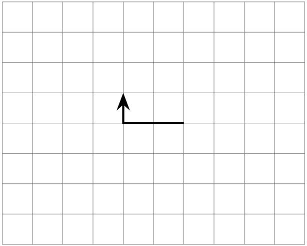

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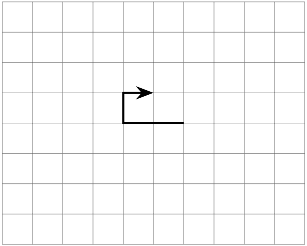

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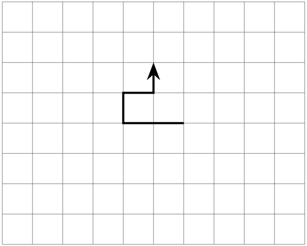

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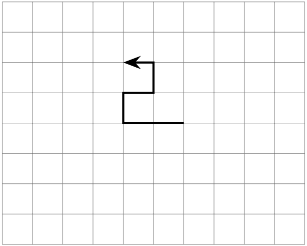

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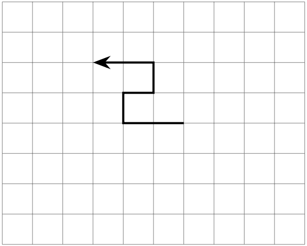

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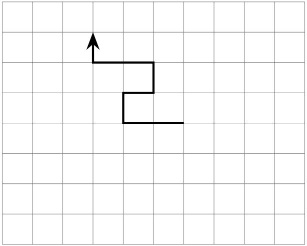

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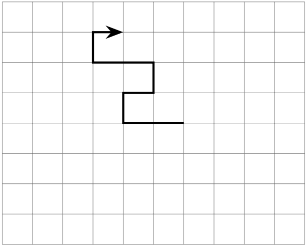

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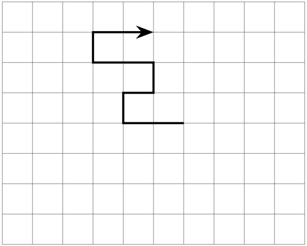

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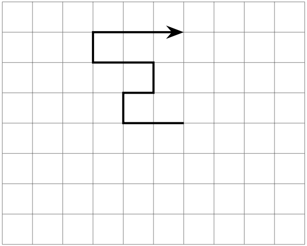

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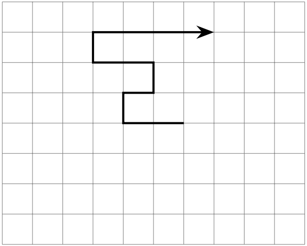

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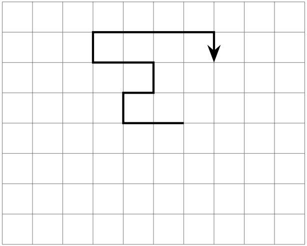

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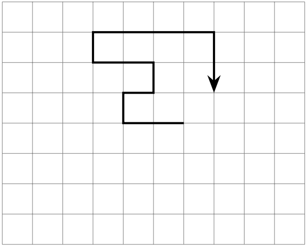

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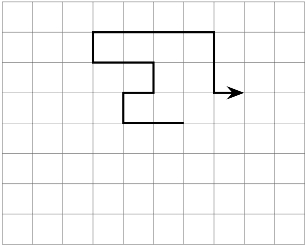

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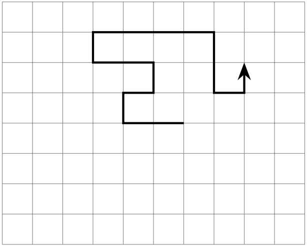

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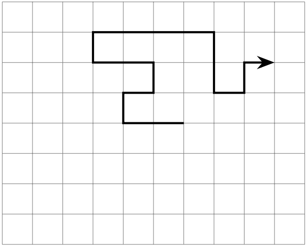

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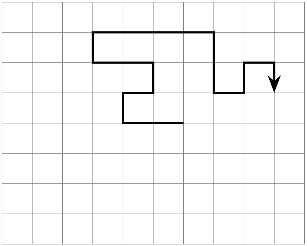

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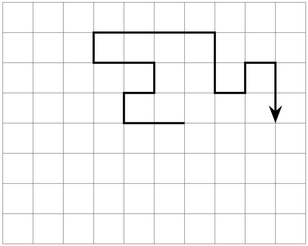

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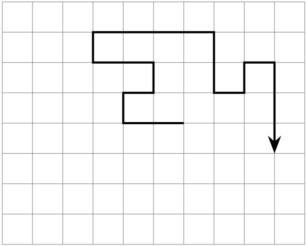

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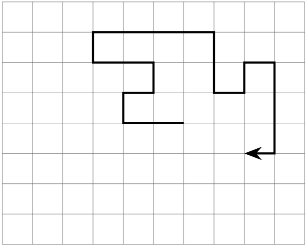

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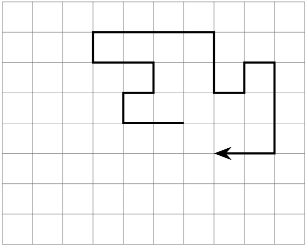

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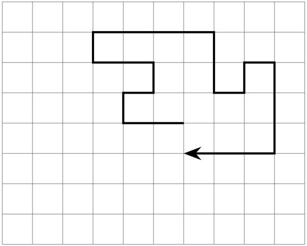

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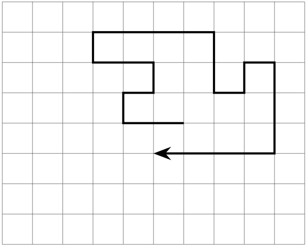

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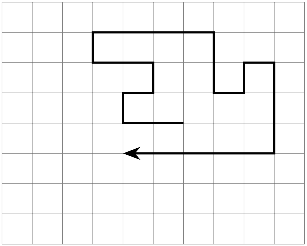

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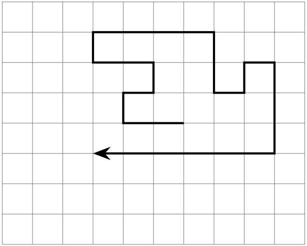

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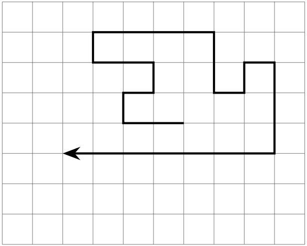

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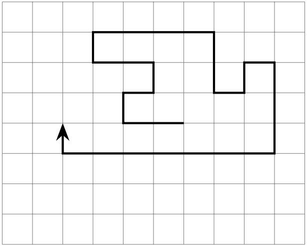

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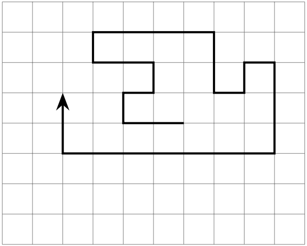

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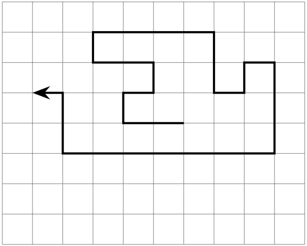

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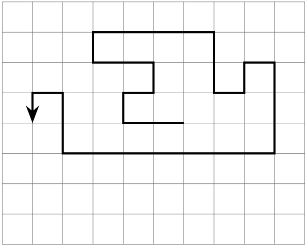

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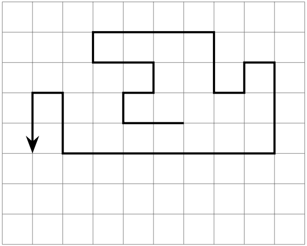

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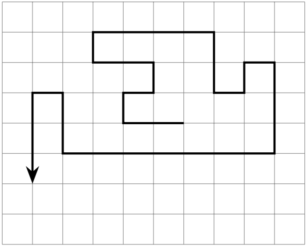

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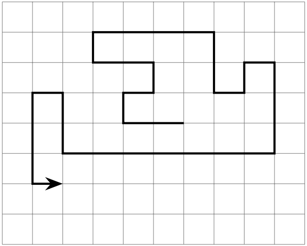

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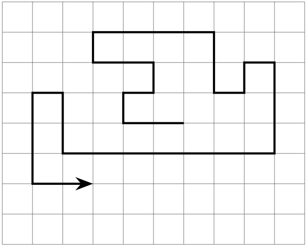

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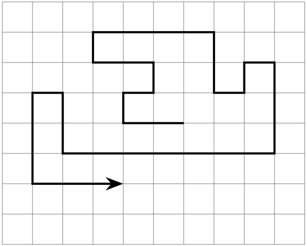

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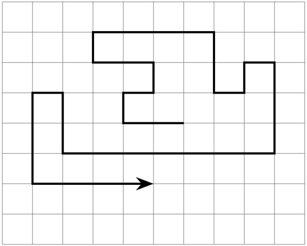

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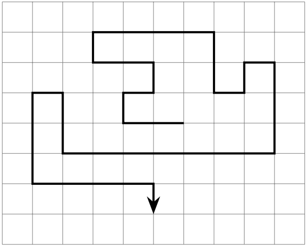

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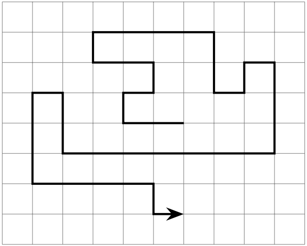

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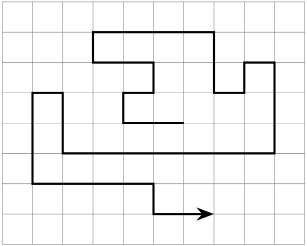

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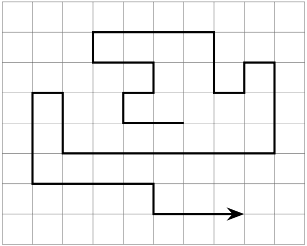

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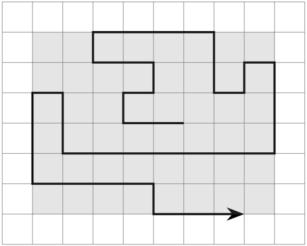

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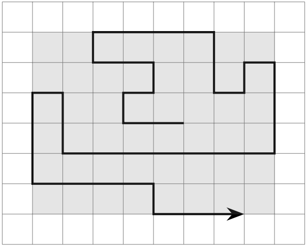

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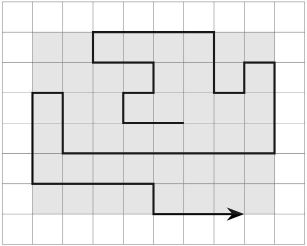

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1-sided prudent SAWs are also PDWs. 2- and 3-sided have been solved (Duchi 2005, Bousquet-Mélou 2010), 4-sided remains unsolved.

Prudent walks

Theorem (Duchi 2005, Bousquet-Mélou 2010)

The numbers of 2- and 3-sided prudent walks are asymptotically

$$c_n^{\text{2-pru}} \sim A_2 \kappa^n \qquad \text{and} \qquad c_n^{\text{3-pru}} \sim A_3 \kappa^n$$

where $\kappa \approx 2.48119$ *is the root of a cubic polynomial and* A_2, A_3 *are positive constants.*

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Conjecture (Dethridge & Guttmann 2008)

The number of 4-sided prudent walks is asymptotically

$$c_n^{\text{4-pru}} \sim A_4 \kappa^n$$

for some positive constant A_4.

Perimeter walks

Can generalise prudent walks by maintaining the bounding box condition while relaxing the prudent condition, to get perimeter walks. Then 2-sided perimeter walks are solvable.

Theorem (B. 2012 (PhD Thesis))

The number of 2-sided perimeter walks is asymptotically

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Self-avoiding bridges

To take things further, we need another definition.

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A (2d) self-avoiding bridge is a SAW whose

- first vertex has strictly minimal y-coordinate
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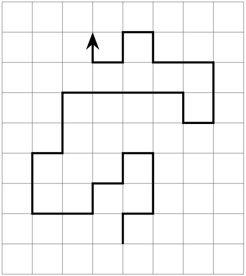

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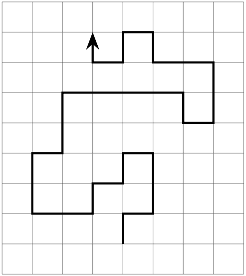

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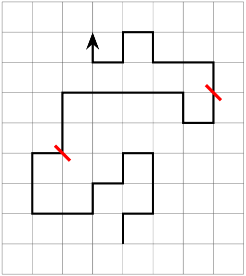

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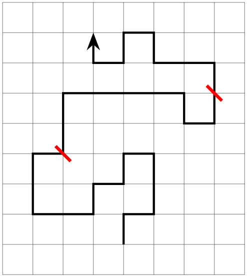

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Note that we do not consider the empty walk to be a bridge.

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This idea can be exploited to get larger solvable classes: Define a class of walks (bridges) whose irreducible components satisfy some set of properties, which allow them to be solved.

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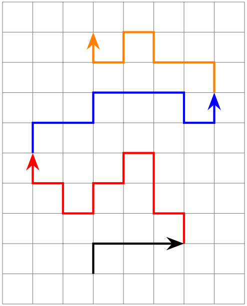

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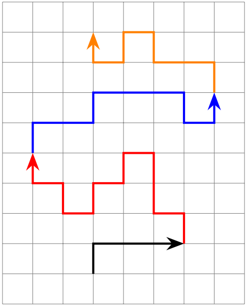

If $I^{\text{PDW}}(z)$ is the generating functions of irreducible partially directed bridges, then

$$B^{\text{WD}}(z) = \frac{I^{\text{PDW}}(z)}{1 - I^{\text{PDW}}(z)}.$$

Weakly directed walks

$I^{\mathrm{PDW}}(z)$ has been solved (Bacher & Bousquet-Mélou 2011).

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The number of weakly directed bridges is asymptotically

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Can we do better?

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An s-sided weakly prudent bridge is a bridge whose irreducible bridge components are s-sided prudent or co-prudent (prudent in the reverse direction) walks, or reflections/rotations thereof.

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1-sided are weakly directed.

We solve the 2-sided case.

3-sided and 4-sided are the same thing, but remain unsolved.

Functional equations

Prudent bridges are a special case of positive prudent walks: those whose first vertex has strictly minimal y-coordinate (upper half plane).

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- distance i from endpoint to NE corner of box
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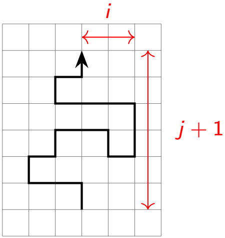

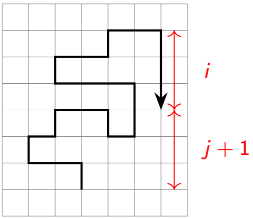

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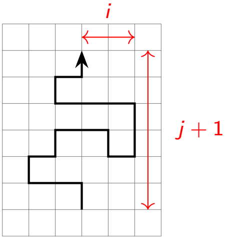

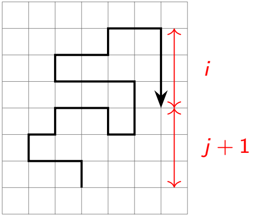

Define $N^{+}(z;u,v)$ and $E^{+}(z;u,v)$ to be the generating functions for those positive walks ending on the N or E side, with u conjugate to i and v conjugate to j. The variables u and v are catalytic.

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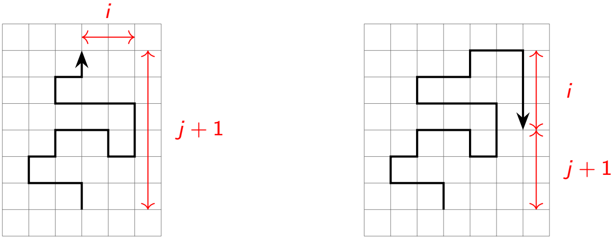

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The bridges are those counted by N^+.

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Do something similar to PDWs (more complicated!) to get

$$\left(1 - \frac{zu}{u - zv} - \frac{z^2 u}{v - zu}\right) E^+(z;u,v) = z - \frac{z^2 v}{u - zv} E^+(z;zv,v) - \frac{z^2 u}{v - zu} E^+(z;u,zu) + zvN^+(z;z,v)$$

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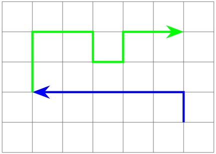
The generating function of 2-sided prudent bridges is $N^+(z; 1, 1)$. We want to get at irreducible bridges. This is not so obvious – the irreducible components of a prudent bridge must be prudent, but if we concatenate two prudent bridges the result may not be prudent:



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Then manipulate some generating functions...

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The first few terms of the series for the irreducible objects are

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Our generating function is then

$$B^{\text{2-WP}}(z) = \frac{I^{\text{2-WP}}(z)}{1 - I^{\text{2-WP}}(z)}$$

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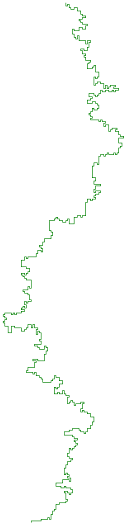
Similarly to 3-sided prudent and weakly directed:

Conjecture (Bacher & B. 2014)

The generating function $B^{2\text{-}WP}(z)$ *of weakly prudent bridges is not D-finite.*

Further results & future work

Can generate random weakly prudent bridges with a critical Boltzmann sampler.



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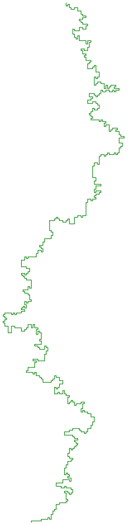
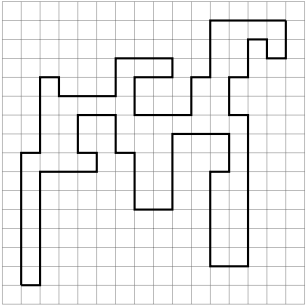
Further generalisations? Solving walk models with more than 2 catalytic variables is often very difficult. (eg. the 3/4-sided prudent or 2-sided perimeter versions would require 3 catalytic variables).



Self-avoiding polygons

A (unrooted, undirected) self-avoiding polygon is a closed loop on the lattice.



It is known that these have the same growth rate as SAWs. If p_{2n} is the number of polygons of perimeter $2n$, then

Theorem (Hammersley 1961)

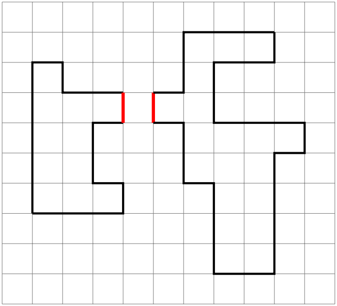
The limit

$$\lim_{n\to\infty} p_{2n}^{1/2n}$$

exists and is equal to μ*, the growth rate of SAWs on the lattice.*

Concatenating polygons

Polygons can be freely concatenated, like self-avoiding bridges. We identify the highest edge on the right side of one polygon with the lowest edge on the left side of the other, and delete them both:



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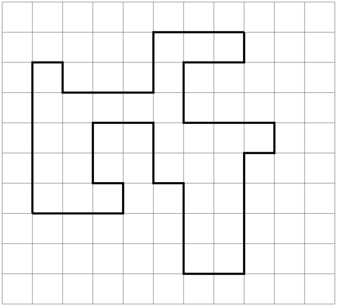
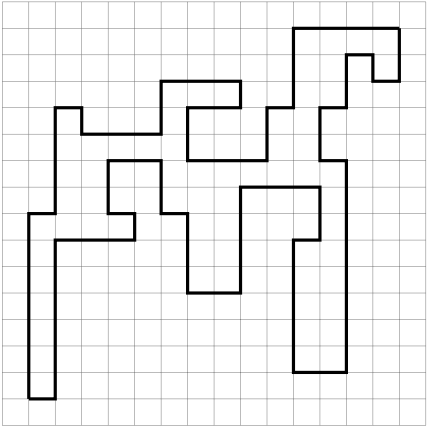


Irreducible polygons

Just as with bridges, we can then define an irreducible polygon to be one that cannot be written as the concatenation of two smaller polygons. Every polygon then has a unique factorization into irreducible components:



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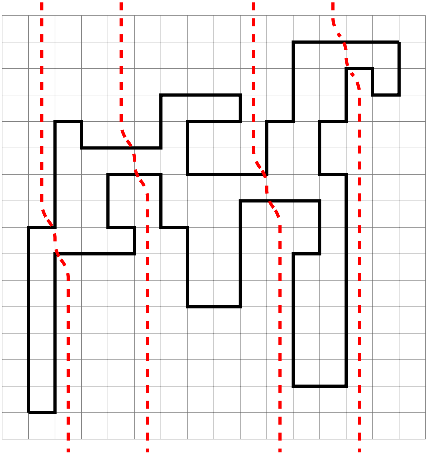

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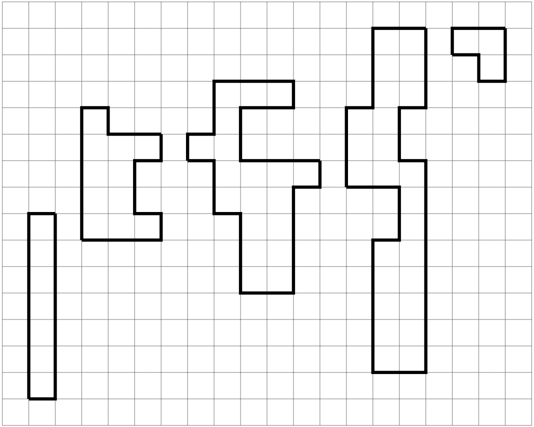

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Still a work in progress!

Reference

Bacher & B., *Weakly prudent self-avoiding bridges*, Proceedings of FPSAC 2014 (Chicago, USA), 827-838.

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Thank you!