Pattern-avoiding permutations: enumeration, asymptotics and generating functions

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Introduction

Knuth 1968 (TAOCP Vol. 1): Which permutations can be sorted using a (infinite capacity) stack?

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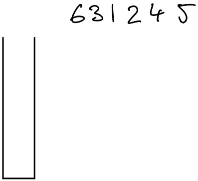

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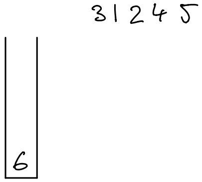

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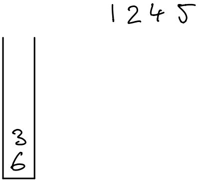

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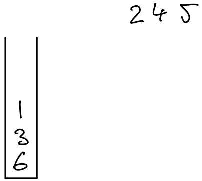

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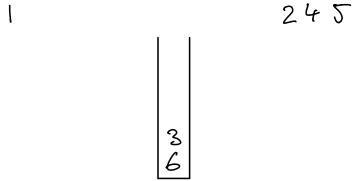

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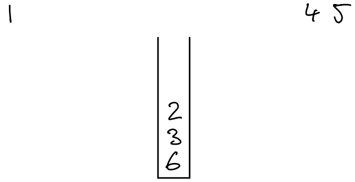

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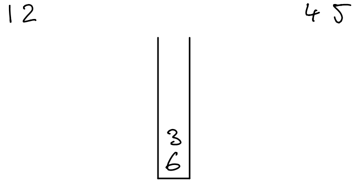

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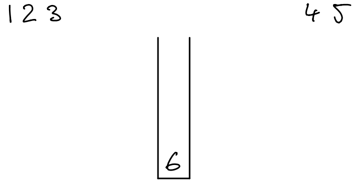

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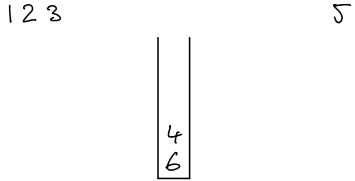

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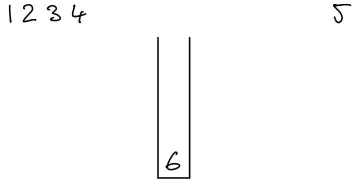

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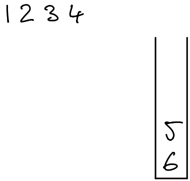

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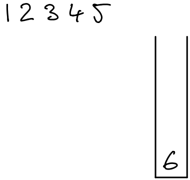

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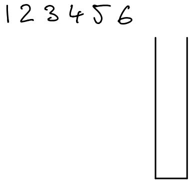

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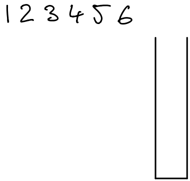

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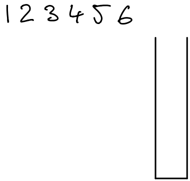

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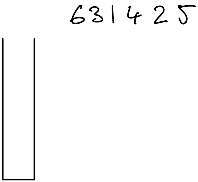

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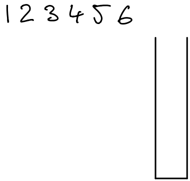

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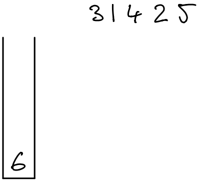

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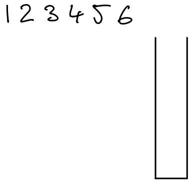

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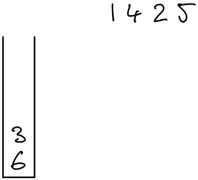

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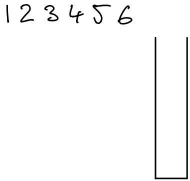

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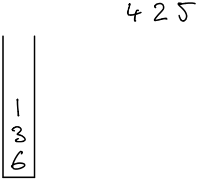

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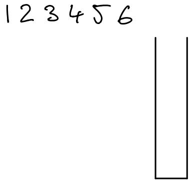

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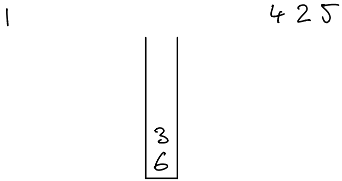

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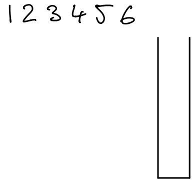

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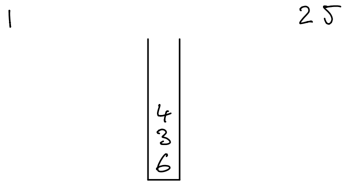

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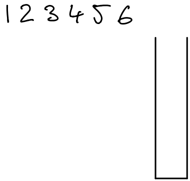

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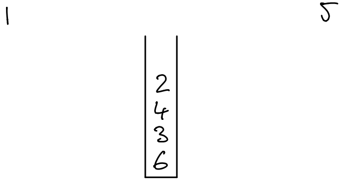

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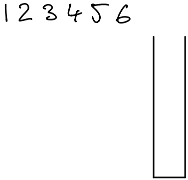

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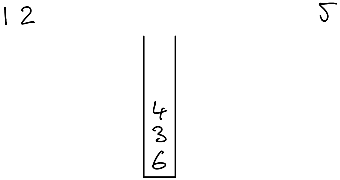

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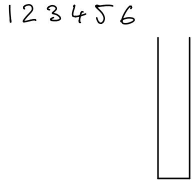

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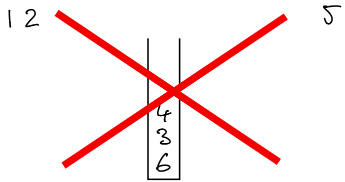

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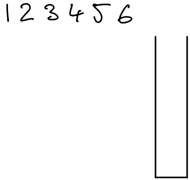

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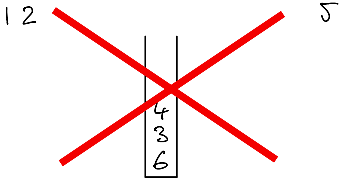

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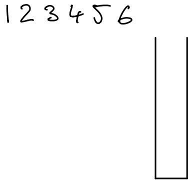

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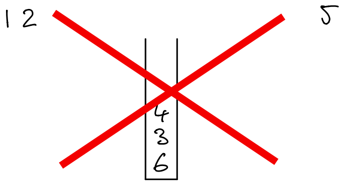

The problem is the 342 subsequence. In fact if any permutation σ has a length-3 subsequence $\sigma_i, \sigma_j, \sigma_k$ with $i < j < k$ and $\sigma_k < \sigma_i < \sigma_j$, it cannot be sorted using a stack.

Definition

Let $\tau \in S_m$ and $\sigma \in S_n$ be permutations with $m \leq n$. Then σ contains pattern τ if there is a subsequence $\sigma_{i_1}, \ldots, \sigma_{i_m}$ in the same relative order as τ. If σ does not contain pattern τ, then σ avoids τ, or is τ-avoiding.

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In fact the converse is also true:

Theorem (Knuth 1968)

A permutation $\sigma \in S_n$ is stack-sortable if and only if σ avoids pattern 231.

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Proof.

By induction. Trivially true if $n = 0, 1, 2$, so let $n \geq 3$ and say σ avoids 231.

Let i be such that $\sigma_i = n$. Then if $j < i < k$, must have $\sigma_j < \sigma_k$.

Now $\sigma' = \sigma_1 \ldots \sigma_{i-1}$ and $\sigma'' = \sigma_{i+1} \ldots \sigma_n$ are 231-avoiding. So sort σ', then push $\sigma_i = n$ onto the stack, then sort σ'', and pop σ_i from the stack. $\square$

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- What is $a_n(\tau) = |\,\mathrm{Av}_n(\tau)|$? As $n \to \infty$?
- Connections between $\mathrm{Av}(\tau)$ and $\mathrm{Av}(\tau')$ for $\tau \neq \tau'$?
- What does a random $\sigma \in \mathrm{Av}(\tau)$ "look like"?

Applications

Theorem (Knuth 1973)

$$a_n(\tau) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

for all $\tau \in S_3$.

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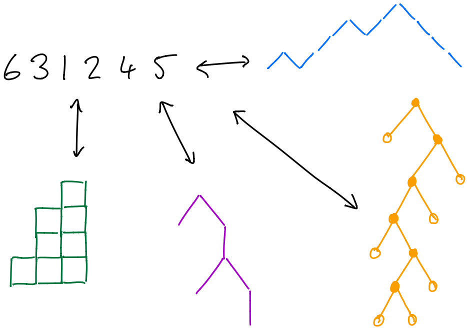
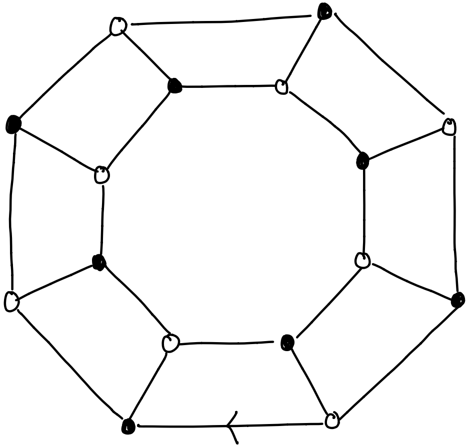
The Catalan numbers count many different objects in combinatorics, including Dyck paths, binary trees, plane trees and parallelogram polyominoes:



Theorem (Bóna 1997)

$\mathrm{Av}_n(1342)$ *is in bijection with the set of rooted bicubic planar maps on* $2(n+1)$ *vertices.*



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Theorem (Murphy 2003)

There is an infinite set $\mathcal{T}$ such that σ is sortable by two infinite stacks in series iff $\sigma \in \mathrm{Av}(\mathcal{T})$.

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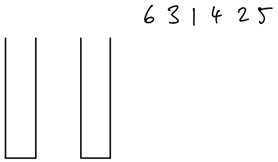

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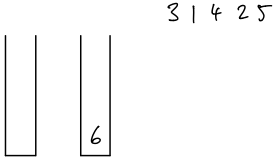

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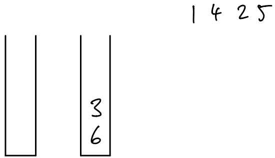

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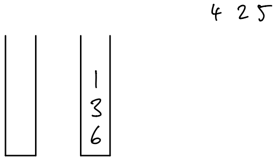

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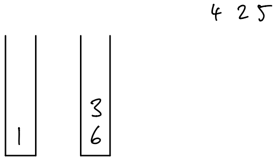

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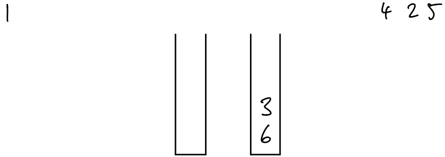

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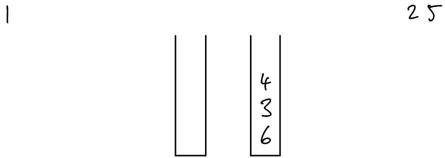

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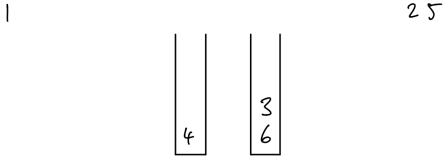

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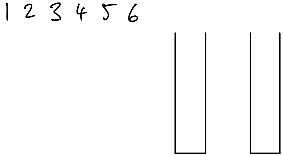
There is an infinite set $\mathcal{T}$ such that σ is sortable by two infinite stacks in series iff $\sigma \in \mathrm{Av}(\mathcal{T})$.



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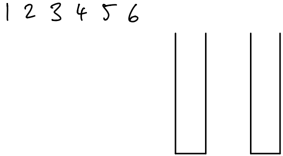

Theorem (Elder 2006)

There a set $\mathcal{T}$ of 20 permutations of size 5, 6, 7 and 8 such that σ is sortable by a stack of depth 2 and an infinite stack in series iff $\sigma \in \mathrm{Av}(\mathcal{T})$.

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Theorem (Bousquet-Mélou & Butler 2007)

A Schubert variety is locally factorial iff its permutation $\sigma \in \mathrm{Av}(1324, 21\bar{3}54)$.

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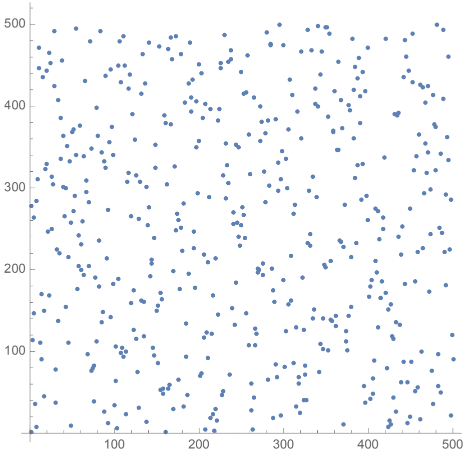

Figure: Random permutation of size 500.

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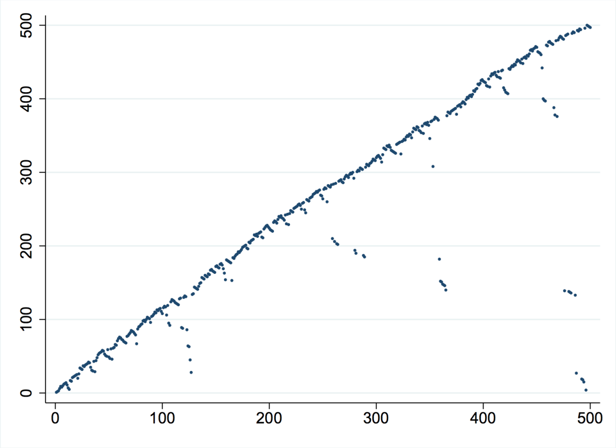

Figure: Random 312-avoiding permutation of size 500. [Madras & Pehlivan, *Random Structures and Algorithms* **49** (2016), pp 599-631.]

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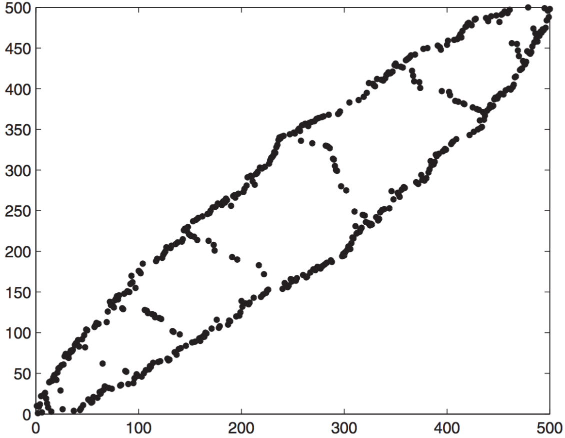

Figure: Random 4231-avoiding permutation of size 500. [Atapour & Madras, *Combinatorics, Probability and Computing* **23** (2014), pp 161–200.]

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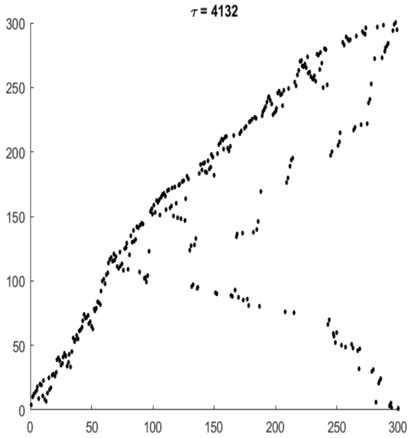

Figure: Random 4132-avoiding permutation of size 300. [Madras & Yildirim, *Electronic Journal of Combinatorics* **24** (2017), #P4.13.]

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But there are some special cases for which this is easier. For example,

Theorem (Chang & Wang 1992)

There is a polynomial time algorithm for the pattern-matching problem for $\mathrm{Av}(123\ldots k)$.

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There is a polynomial time algorithm for the pattern-matching problem for $\mathrm{Av}(2413, 3142)$ *(separable permutations).*

Back to Catalan

Recall $a_n(231) = C_n = \frac{1}{n+1}\binom{2n}{n}$. Why?

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Let $\sigma \in \mathrm{Av}_n(231)$, and condition on the position of n. If $\sigma_i = n$, then $\sigma' = \sigma_1 \ldots \sigma_{i-1} \in \mathrm{Av}_{i-1}(231)$ and $\sigma'' = \sigma_{i+1} \ldots \sigma_n$ (after subtracting $i-1$ from all values) is $\in \mathrm{Av}_{n-i}(231)$.

Conversely, given any $\sigma' \in \mathrm{Av}_{i-1}(231)$ and $\sigma'' \in \mathrm{Av}_{n-i}(231)$, we can construct $\sigma \in \mathrm{Av}_n(231)$ by shifting σ'' up by $i-1$ and concatenating $\sigma' n \sigma''$.

Summing over all possible values of i,

$$a_n(231) = \sum_{i=1}^{n} a_{i-1}(231) a_{n-i}(231)$$

with $a_0(231) = 1$.

This is the same recurrence satisfied by C_n.

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$$a_n(123\ldots k) \sim c\frac{(k-1)^{2n}}{n^{k(k-2)/2}}$$

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Some other results are known, but first we need a new definition...

Wilf equivalence

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For length 5 there are 16 Wilf equivalence classes.

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$$a_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}$$

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Before we get to Av(1324), another digression...

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If F is not entire, then F has a singularity (a point of non-analyticity) at some $z \in \mathbb{C}$ with $|z| = \rho$. This is the dominant singularity (or singularities).

Lemma

If F has a single dominant singularity at $z = \rho$, and $F(z) \sim A(1 - z/\rho)^\alpha$ as $z \to \rho$ with A, α constant and $\alpha \notin \mathbb{N}$, then

$$f_n = \frac{An^{\alpha-1}\rho^{-n}}{\Gamma(\alpha)} \left(1 + O(n^{-1})\right)$$

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F is D-finite if $F(z)$ is the solution to a linear ODE with coefficients in $\mathbb{Z}[z]$.

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D-finite functions are "nice" since their coefficients satisfy finite linear recurrences. For example, the Catalan OGF is algebraic:

$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}$$

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$$(n + 1)C_n - 2(2n - 1)C_{n-1} = 0.$$

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A useful fact:

Lemma

If $f(z)$ is a D-finite function then f has a finite number of singularities in $\mathbb{C}$.

For all consecutive patterns of length two and three, and 16 of the 24 for length four, ODEs for the reciprocals of the EGFs are known [Elizalde & Noy 2003, 2012]. Also for many of length ≥ 5. (Why reciprocals? Stay tuned...)

Led some to conjecture that the reciprocal of the EGF is always D-finite. (Definitely not the case for EGF itself, e.g. for $\sigma = 123$.)

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Proof.

If $P_k(z)f^{(n)}(z)$ is the highest-order term in the ODE satisfied by f, then every singularity of f must be a root of P_k. □

Why reciprocals?

Given pattern σ of length m, a k-cluster of length n w.r.t. σ is a pair (π, S) where

- π is a permutation of length n,
- $S = (s_1 = 1, s_2, \ldots, s_k = n - m + 1)$ is an sequence of indices such that $s_i - s_{i-1} < m$, and
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Theorem (Goulden & Jackson 1983)

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For $\tau = 1423$: In a k-cluster, condition on the first m incidences of τ overlapping by two positions:

$$r_{n,k} = \sum_{4 \leq 2m+2 \leq n} \binom{n-m-2}{m} r_{n-2m-1,k-m}.$$

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2. Show that each of the denominators has a distinct root in $\mathbb{C}$
3. Show that each root is not cancelled by the numerator $\rightarrow$ genuine pole

Generalisation to longer patterns

The argument generalises to consecutive patterns of the form

$$1m23\ldots(m-2)(m-1).$$

$\Rightarrow$ Infinite family of consecutive patterns for which the reciprocal of the EGF is not D-finite.

Reference

NRB, A R Conway & A J Guttmann: On consecutive pattern-avoiding permutations of length 4, 5 and beyond, *Discrete Mathematics and Theoretical Computer Science* **19** (2018), Article #8.

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Thank you!