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The enumeration of prudent polygons by area and its unusual asymptotics

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ABSTRACT

Prudent walks are special self-avoiding walks that never take a step towards an already occupied site, and *k-sided prudent walks* (with $k = 1, 2, 3, 4$) are, in essence, only allowed to grow along k directions. Prudent polygons are prudent walks that return to a point adjacent to their starting point. Prudent walks and polygons have recently been enumerated by length and perimeter by Bousquet-Mélou and Schwerdtfeger. We consider the enumeration of *prudent polygons* by *area*. For the 3-sided variety, we find that the generating function is expressed in terms of a q-hypergeometric function, with an accumulation of poles towards the dominant singularity. This expression reveals an unusual asymptotic structure of the number of polygons of area n, where the critical exponent is the transcendental number $\log_2 3$ and the amplitude involves tiny oscillations. Based on numerical data, we also expect similar phenomena to occur for 4-sided polygons. The asymptotic methodology involves an original combination of Mellin transform techniques and singularity analysis, which is of potential interest in a number of other asymptotic enumeration problems.

1. Introduction

The problem of enumerating *self-avoiding walks* (SAWs) and *polygons* (SAPs) on a lattice is a famous one, whose complete solution has thus far remained most elusive. For the square lattice, it is conjectured that the number SAW_n of walks of *length* n and the number SAP_n of polygons of *perimeter* n each satisfy an asymptotic formula of the general form

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$$C \cdot \mu^n \cdot n^\beta, \tag{1}$$

where $C, \mu \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$. (In the case of polygons, it is understood that n must be restricted to even values.) The number μ is the "growth constant" and the number β is often referred to as the "critical exponent". More precisely, the following expansions are conjectured,

$$\mathrm{SAW}_n \underset{n\to\infty}{\sim} C_1 \cdot \mu^n \cdot n^{11/32}, \qquad \mathrm{SAP}_n \underset{n\to\infty}{\sim} C_2 \cdot \mu^n \cdot n^{-5/2}, \tag{2}$$

for some $C_1, C_2 > 0$.

For the square lattice, numerical methods based on acceleration of convergence and differential approximants suggest the value $\mu = 2.6381585303\ldots$. This estimate is indistinguishable from the solution of the biquadratic equation $13\mu^4 - 7\mu^2 - 581 = 0$, which we consider to be a useful mnemonic. This was observed by Conway, Enting and Guttmann [6] in 1993, and verified to 11 significant digits by Jensen and Guttmann [23] in 2000, based on extensive numerical analysis of the sequence (SAP_{2n}) up to $n = 45$. (Remarkably enough, for the honeycomb lattice, it had long been conjectured that the growth constant of walks is the biquadratic number $\sqrt{2+\sqrt{2}}$, a fact rigorously established only recently by Duminil-Copin and Smirnov [13].)

As regards critical exponents, the conjectured value $\beta = \frac{11}{32}$ for walks is supported by results of Lawler, Schramm, and Werner that relate the self-avoiding walk to the "stochastic Loewner Evolution" (SLE) process of index 8/3; see, for instance, the account in Werner's inspiring lecture notes [37]. For (unrooted) polygons, the value $\beta = -\frac{5}{2}$ was suggested by numerical analysis of the exact counting sequence, with an agreement to the seventh decimal place [23]. It is also supported by the observation that many simplified, exactly solvable, (naturally rooted) models of self-avoiding polygons appear to exhibit an $n^{-3/2}$ universal behavior — for these aspects, we refer to the survey by Bousquet-Mélou and Brak [4], as well as the books [15,20,32].

Regarding lattice polygons, which are closed walks, there is also considerable interest in enumeration according to *area*, rather than perimeter. There are at least two reasons for this. Enumerating polygons by both perimeter and area provides a very natural and powerful model of vesicles, with considerable biological interest [32]. Secondly, it is closely related to the classical unsolved problem of enumerating polyominoes (also known as animals), according to the number of cells they contain. Polygons are a proper subset of polyominoes, so any results obtained on polygons enumerated by area may illuminate the polyomino enumeration problem.

Conjecturally [24], the number a_n (respectively, b_n) of polygons (respectively, polyominoes) comprised of n cells satisfies asymptotic estimates of the form

$$a_n \underset{n\to\infty}{\sim} C_3 \cdot (3.9709\ldots)^n \cdot n^{-1}, \qquad b_n \underset{n\to\infty}{\sim} C_4 \cdot (4.0625\ldots)^n \cdot n^{-1},$$

for some $C_3, C_4 \in \mathbb{R}_{>0}$; these asymptotic estimates are still of the form (1), with the critical exponent $\beta = -1$. Interestingly enough, the critical exponent $\beta = 0$, corresponding to a simple pole of the associated generating function, is otherwise known to arise in several simplified models, such as column-convex, convex, and directed polyominoes [2,4,20].

As the foregoing discussion suggests, there is considerable interest in solving, exactly, probabilistically, or asymptotically, restricted models of self-avoiding walks and polygons. Beyond serving to develop informed conjectures regarding more complex models, this is relevant to areas such as statistical physics and the statistical mechanics of polymers [32]. For combinatorialists, we may observe that consideration of such models has served as a powerful incentive to develop new counting methods based on generating functions [4,7,20,36], including transfer matrix methods and what is known as the "kernel method".

The present article focuses on a special type of self-avoiding polygon, the 3-*sided prudent polygon* (to be defined shortly — see Definition 1 in Section 2), when these are enumerated according to area. Roughly, a walk is prudent if it never takes a step towards an already occupied site and it is 2-, 3-, 4-sided if it has, respectively, 2, 3, or 4 allowed directions of growth; a prudent polygon is a prudent walk that is almost closed. For area n, we will obtain a precise asymptotic formula (Theorem 2 below),

$$PA_n \sim C(n) \cdot 2^n \cdot n^g, \tag{3}$$

one that has several distinguishing features: (i) the critical exponent is the *transcendental number* $g = \log_2 3$, in sharp contrast with previously known examples where it is invariably a "small" rational number; (ii) the multiplier C is no longer a constant, but a bounded quantity that *oscillates* around the value $0.10838\ldots$ and does so with a minute amplitude of 10^{-9}. The oscillations cannot be revealed by any standard numerical analysis of the counting sequence PA_n, but such a phenomenon may well be present in other models, and, if so, it could change our whole view of the asymptotic behavior of such models.

Plan and results of the paper. Prudent walks are defined in Section 2, where we also introduce the 2-sided, 3-sided, and 4-sided varieties. The enumeration of 2- and 3-sided walks and polygons by *perimeter* is the subject of insightful papers by Bousquet-Mélou [3] and Schwerdtfeger [33] who obtained both exact generating function expressions and precise asymptotic results. In Sections 2.2–2.4, we provide the algebraic derivation of the corresponding area results: the enumeration of 2- *and* 3-*sided prudent polygons according to area* is treated there; see Theorem 1 for our first main result. For completeness, we also derive a functional equation for the generating function of 4-sided prudent polygons (according to area), which parallels an incremental construction of [3, §6.5] — this functional equation suffices to determine the counting sequence in polynomial time. Section 3, dedicated to the asymptotic analysis of the number of 3-sided prudent polygons, constitutes what we feel to be the main contribution of the paper. We start from a q-hypergeometric representation of the generating function of interest, $PA(z)$, and proceed to analyze its singular structure: it is found that $PA(z)$ has poles at a sequence of points that accumulate geometrically fast to $\frac{1}{2}$; then, the Mellin transform technology [17] provides access to the asymptotic behavior of $PA(z)$, as $z \to \frac{1}{2}$ in extended regions of the complex plane. Singularity analysis [20, Chapters VI–VII] finally enables us to determine the asymptotic form of the coefficients PA_n (see Theorem 2) and even derive a complete asymptotic expansion (Theorem 3). As already mentioned, the non-standard character of the asymptotic phenomena found is a distinctive feature. Section 4 concludes the paper with brief remarks relating to asymptotic methodology. In particular, experiments suggest that similar asymptotic phenomena are likely to be encountered in the enumeration of 4-sided prudent polygons.

A preliminary announcement of the results of the present paper is the object of the communication [1].

2. Prudent walks and polygons

One interesting sub-class of self-avoiding walks (SAWs) for which a number of exact solutions have been recently found are *prudent* walks. Introduced by Préa [30], these are SAWs which never take a step towards an already occupied node. Exact solutions of prudent walks on a 2-dimensional square lattice were later studied by Duchi [11] and Bousquet-Mélou [3], who were able to enumerate certain sub-classes. The enumeration of the corresponding class of polygons is due to Schwerdtfeger [33]. In this section, we first recall the definition of prudent walks and polygons, then summarize the known results of [3,33] relative to their enumeration by length or perimeter; see Section 2.1, where 2-, 3-, and 4-sided prudent walks are introduced. We then examine the enumeration of polygons according to area, in each of the three non-trivial cases. The case of 2-sided polygons is easy enough (Section 2.2). The main result of this section is Theorem 1 of Section 2.3, which provides an explicit generating function for 3-sided polygons — it is on this expression that our subsequent asymptotic treatment is entirely based. In the case of 4-sided (i.e., "general") polygons, we derive in Section 2.4 a system of functional equations that determines the generating function and amounts to a polynomial-time algorithm for the generation of the counting sequence.

2.1. Main definitions and results

We use the same classification scheme as the authors of [3,33]. By definition, the endpoint of every prudent walk always lies on the boundary of the smallest lattice rectangle which contains the entire walk, referred to here as the *bounding box* or just *box*. This property leads to a natural classification of prudent walks (see Fig. 1).

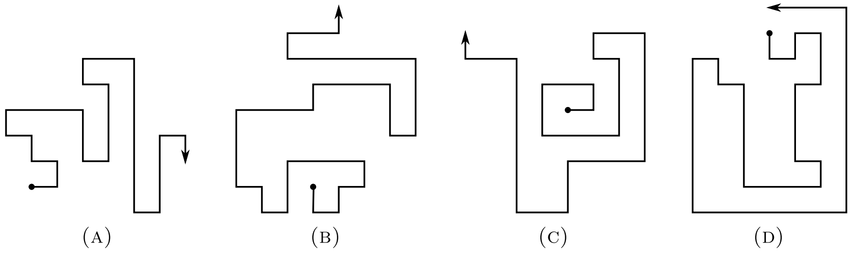

Fig. 1. Examples of (A) a two-sided prudent SAW; (B) a three-sided prudent SAW; (C) an (unrestricted) prudent SAW; (D) a prudent SAW leading to a prudent SAP.

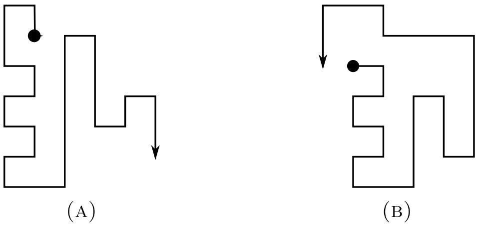

Fig. 2. Examples of the (A) prudent walks and (B) prudent polygons which we exclude from the definition of 3-sided.

Definition 1. Let ω be a prudent walk of length n, and let ω_i be the prudent walk comprising the first i steps of ω. Let b_i be the bounding box of ω_i. Then ω is 1-*sided* if ω_i ends on the north side of b_i for each $i = 0, 1, \ldots, n$; 2-*sided* if each ω_i ends on the north or east sides of b_i; 3-*sided* if each ω_i ends on the north, east or west sides of b_i (with one caveat, described below); and 4-*sided* (or *unrestricted*) if each ω_i may end on any side of b_i.

Remark. The issue with 3-sided prudent walks is encapsulated by the walk $(0,0) \to (1,0) \to (1,-1) \to (0,-1)$. If one draws the walk's box after each (discrete) step, then it is clear that the walk always ends on the north, east or west sides, seemingly fulfilling the 3-sided requirement. However, if the walk is taken to be continuous, then along the step $(1,-1) \to (0,-1)$, the endpoint is only on the south side. In general this occurs when a walk steps from the south-east corner of its box to the south-west corner (or vice versa) when the box has width one and non-zero height. Allowing such walks forces us to account for structures like those in Fig. 2; while this is certainly possible, it complicates a number of rational terms and contributes little to the asymptotic behavior of the model. For this reason we follow the examples of Bousquet-Mélou and Schwerdtfeger [3,33] and exclude these cases.

An equivalent definition of the 3-sided walks considered here is to forbid two types of steps: when the box has non-zero width, a south step may not be followed by a west (resp. east) step when the walk is on the east (resp. west) side of the box.

Walks (length). The enumeration of 1-sided prudent walks (also known as *partially directed walks*) is straightforward, and the generating function for such walks is rational — we will not discuss these any further. Duchi [11] successfully found the generating function for 2-sided walks, showing it to be algebraic. Bousquet-Mélou [3] solved the problem of 3-sided prudent walks, finding the generating function to be non-D-finite. For the unrestricted case, functional equations were found by both Duchi and Bousquet-Mélou, but at present these equations remain unsolved.

The dominant singularity of the generating functions for 2-sided and 3-sided prudent walks is a simple pole at $\rho = 0.4030317168\ldots$, the smallest real root of $1 - 2x - 2x^2 + 2x^3$. Dethridge and Guttmann [9] conjecture that the same is true for unrestricted prudent walks, based on a computer-generated series of 100 terms. They also conjectured that the generating function for unrestricted prudent walks is non-holonomic[1] (non-D-finite). Here is a summary of known results, with the estimates tagged with a question mark being conjectural ones:

Prudent walks	Generating function	Asymptotic number	References
2-sided	algebraic	$\kappa_2 \cdot \rho^{-n}$, $\rho^{-1} \simeq 2.481$	Duchi [11], Bousquet-Mélou [3]
3-sided	non-holonomic	$\kappa_3 \cdot \rho^{-n}$, $\rho^{-1} \simeq 2.481$	Bousquet-Mélou [3]
4-sided	functional equation	$\kappa_4 \cdot \rho^{-n}$, $\rho^{-1} \simeq 2.481$ (?)	Dethridge and Guttmann [9].

(4)

The values of the multipliers, after [9], are $\kappa_2 = 2.51\ldots$, $\kappa_3 = 6.33\ldots$ and (estimated) $\kappa_4 \approx 16.12$.

Prudent polygons (perimeter). Self-avoiding polygons (SAPs) are self-avoiding walks which end at a node adjacent to their starting point (excluding walks of a single step). If the walk has length $n-1$ then the polygon is said to have *perimeter n*.

Definition 2. A *prudent* self-avoiding polygon (prudent polygon) is a SAP for which the underlying SAW is prudent. In the same way, a prudent polygon is 1-sided (resp. 2-sided, etc.) if its underlying prudent walk is 1-sided (resp. 2-sided, etc.).

A 1-sided prudent polygon starting at $(0,0)$ must end at $(0,1)$, thus consisting only of a single row of cells and having a rational generating function. The enumeration of 2- and 3-sided prudent polygons by perimeter has been addressed by Schwerdtfeger [33]. The non-trivial 2-sided prudent polygons are essentially inverted bargraphs [31], and so the 2-sided case has an algebraic generating function. Schwerdtfeger finds the 3-sided prudent polygons to have a non-D-finite generating function.

If $PP^{(k)}(z) = \sum_{n\geqslant 0} p_n^{(k)} z^n$ is defined to be the half-perimeter generating function for k-sided prudent polygons (so $p_n^{(k)}$ is the number of k-sided prudent polygons with perimeter $2n$), then the following holds [33]: (i) the dominant singularity of $PP^{(2)}(z)$ is a square root singularity at $\sigma = 0.2955977\ldots$, the unique real root of $1-3x-x^2-x^3$. So $p_n^{(2)} \sim \lambda_2 \sigma^{-n} n^{-3/2}$ as $n \to \infty$, where λ_2 is a constant; (ii) the dominant singularity of $PP^{(3)}(z)$ is a square root singularity at $\tau = 0.24413127\ldots$, where τ is the unique real root of $\tau^5+6\tau^3-4\tau^2+17\tau-4$. So $p_n^{(3)} \sim \lambda_3 \tau^{-n} n^{-3/2}$ as $n \to \infty$, where λ_3 is a constant. Schwerdtfeger has furthermore classified 4-sided prudent polygons in such a way as to allow for functional equations in the generating functions to be written. Unfortunately no one has thus far been able to obtain a solution from said equations. We again present a summary of known results:

Polygons (perimeter)	Generating function	Asymptotic number	References
2-sided	algebraic	$\lambda_2 \cdot \sigma^{-n} n^{-3/2}$, $\sigma^{-1} \simeq 3.382$	Schwerdtfeger [33]
3-sided	non-holonomic	$\lambda_3 \cdot \tau^{-n} n^{-3/2}$, $\tau^{-1} \simeq 4.096$	Schwerdtfeger [33]
4-sided	functional equation	$\lambda_4 \cdot \upsilon^{-n} n^{-\delta}$, $\upsilon^{-1} \approx 4.415$ (?)	Dethridge et al. [8]

(5)

The empirical estimates regarding 4-prudent polygons are taken from [8]. They are somewhat imprecise, and it is suspected that the critical exponent satisfies $\delta = -3.5 \pm 0.1$, with $\delta = -7/2$ a compatible value.

[1] A function is said to be *holonomic* or *D-finite* if it is the solution to a linear differential equation with polynomial coefficients; see [34, Chapter 6] and [20, Appendix B.4].

Prudent polygons (area). The focus of this paper is on the enumeration of prudent polygons by *area*, rather than perimeter. The constructions we use here are essentially the same as Schwerdtfeger's [33]; the resulting functional equations and their solutions, however, turn out to be quite different, as will be revealed by the peculiar singularity structure of the generating functions and the non-trivial asymptotic form of the coefficients. We have modified Schwerdtfeger's construction for 4-sided prudent polygons slightly to allow for an easier conversion into a recursive form (see Section 2.4).

We will denote the area generating function for k-sided prudent polygons by $PA^{(k)}(q) = \sum_{n\geqslant 1} PA_n^{(k)} q^n$. For 3- and 4-sided prudent polygons, it is necessary to measure more than just the area — in these cases, additional *catalytic* variables will be used (see [3] for a more thorough explanation).

2.2. Enumeration of 2-sided polygons by area

The non-trivial 2-sided prudent polygons can be constructed from bargraphs. Let $B(q) = \sum_{n\geqslant 1} b_n q^n$ be the area generating function for these objects. The area generating function for bargraphs, $B(q)$, is

$$B(q) = \frac{q}{1-2q}$$

and so $b_n = 2^{n-1}$ for $n \geqslant 1$. (Bargraphs are a graphical representation of integer compositions.)

Proposition 1. *The area generating function for* 2*-sided prudent polygons is*

$$PA^{(2)}(q) = \frac{2q}{1-2q} + \frac{2q}{1-q},$$

and so the number of such polygons is $PA_n^{(2)} = 2^n + 2$ *for* $n \geqslant 1$.

Proof. A 2-sided prudent polygon must end at either $(0,1)$ or $(1,0)$. Reflection in the line $y = x$ will not invalidate the 2-sided property, so it is sufficient to enumerate those polygons ending at $(1,0)$ and then multiply the result by two.

The underlying 2-sided prudent walk cannot step above the line $y = 1$, nor to any point (x, y) where $x, y < 0$. So any polygon beginning with a west step must be a single row of cells to the left of the y-axis. The generating functions for these polygons is then $q/(1-q)$.

A polygon starting with a south or east step must remain on the east side of its box until it reaches the line $y = 1$, at which point it has no choice but to take west steps back to the y-axis. It can hence be viewed as an upside-down bargraph with north-west corner $(0,1)$. The area generating function for these objects is $B(q) = q/(1-2q)$.

Adding these two possibilities together and doubling gives the result. □

2.3. Enumeration of 3-sided polygons by area

When constructing 3-sided prudent polygons, we will use a single catalytic variable which measures width. To do so we will need to measure bargraphs by width. Let $B(q,u) = \sum_{n\geqslant 1}\sum_{i\geqslant 1} b_{n,i} q^n u^i$ be the area-width generating function for bargraphs (so $b_{n,i}$ is the number of bargraphs with area n and width i).

The area-width generating function for bargraphs, $B(q,u)$, satisfies the equation

$$B(q,u) = \frac{qu}{1-q} + \frac{qu}{1-q} B(q,u), \tag{6}$$

which is obtained by successively adding columns. Accordingly, by solving the functional equation, we obtain

$$B(q,u) = \frac{qu}{1-q-qu}$$

and so $b_{n,i} = \binom{n-1}{i-1}$ for $n, i \geqslant 1$. (Clearly, $b_{n,i}$ counts compositions of n into i summands.)

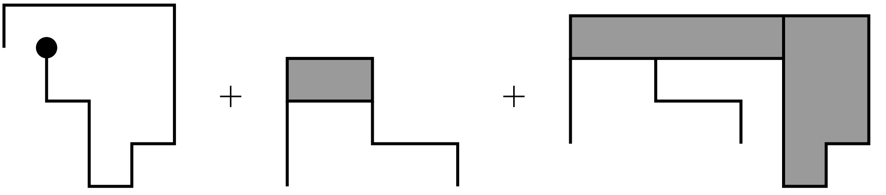

Fig. 3. The decomposition used to construct 3-sided prudent polygons.

Let $W(q,u) = \sum_{n\geqslant 1}\sum_{i\geqslant 1} w_{n,i}q^n u^i$ be the area-width generating function for 3-sided prudent polygons which end at $(-1,0)$ in a counter-clockwise direction. As we will see, this is the most complex type of 3-sided prudent polygon; everything else is either a reflection of this or can be constructed from something simpler. (See Fig. 3.)

Lemma 1. *The area-width generating function for* 3*-sided prudent polygons ending at* $(-1,0)$ *in a counter-clockwise direction,* $W(q,u)$*, satisfies the functional equation*

$$W(q,u) = qu\big(1+B(q,u)\big) + \frac{q}{1-q}\big(W(q,u)-W(q,qu)\big) + qu\big(1+B(q,u)\big)W(q,qu). \tag{7}$$

Proof. The underlying prudent walk cannot step to any point (x,y) with $x, y < 0$, nor to any point with $x < -1$. It must approach the final node $(-1,0)$ from above. So the only time the endpoint can be on the west side of the box and not the north or south is when the walk is stepping south along the line $x = -1$. So prior to reaching the line $x = -1$, the walk must in fact be 2-sided. Note that the north-west corner of the box must be a part of the polygon.

If the walk stays on or below the line $y = 1$, then (as has been seen in Proposition 1), it either reaches the point $(0,1)$ with a single north step, or by forming an upside-down bargraph. This must then be followed by a west step to $(-1,1)$, then a south step. This will form either a single square or a bargraph with a single square attached to the north-west corner, giving the first term on the right-hand side of (7).

Since the north-west corner of the box of any of these polygons is part of the polygon, it is valid to add a row of cells to the top of an existing polygon (so that the west sides line up). This can be done to any polygon. If the new row is not longer than the width of the existing polygon we obtain the term

$$\sum_{n\geqslant 1}\sum_{i\geqslant 1} w_{n,i}q^n u^i \cdot \sum_{k=1}^{i} q^k = q\sum_{n\geqslant 1}\sum_{i\geqslant 1} w_{n,i}q^n u^i \cdot \frac{1-q^i}{1-q},$$

giving the second term in the right-hand side of (7).

Note. For the remainder of this subsection, we will omit unwieldy double or triple sums like the one above, and instead give recursive relations only in terms of the generating functions.

Instead, the new row may be longer than the width of the existing polygon. In this case, as the walk steps east along this new row, it will reach a point at which there are no occupied nodes south of its position, and it will hence be able to step south in a prudent fashion. It must then remain on the east side of the box until reaching the north side, at which point it steps west to $x = -1$ and then south to the endpoint. This effectively means we have added a row of length equal to the width $+1$, and then (possibly) an arbitrary bargraph. So we obtain

$$quW(q,qu)\big(1+B(q,u)\big)$$

which gives the final term in the right-hand side of (7). $\square$

Lemma 2. *The area-width generating function for* 3*-sided prudent polygons ending at* $(-1,0)$ *in a counter-clockwise direction is*

$$W(q,u)=\sum_{m=0}^{\infty}F\big(q,q^m u\big)\prod_{k=0}^{m-1}G\big(q,q^k u\big),$$

where

$$F(q,u)=\frac{qu(1-q)^2}{(1-2q)(1-q-qu)},\qquad G(q,u)=\frac{-q(1-q-u+qu-q^2u)}{(1-2q)(1-q-qu)}.$$

Proof. Substituting $BW(q,u)=qu/(1-q-qu)$ into (7) and rearranging gives

$$W(q,u)=F(q,u)+G(q,u)W(q,qu). \tag{8}$$

Substituting $u\mapsto uq$ gives

$$W(q,qu)=F(q,qu)+G(q,qu)W\big(q,q^2u\big) \tag{9}$$

and combining these yields

$$W(q,u)=F(q,u)+F(q,qu)G(q,u)+G(q,u)G(q,qu)W\big(q,q^2u\big). \tag{10}$$

Repeating for $u\mapsto q^2u,q^3u,\dots,q^Mu$ will give

$$W(q,u)=\sum_{m=0}^{M}F\big(q,q^m u\big)\prod_{k=0}^{m-1}G\big(q,q^k u\big)+\prod_{m=0}^{M}G\big(q,q^m u\big)W\big(q,q^{M+1}u\big). \tag{11}$$

We now seek to take $M\to\infty$. To obtain the result stated in the lemma, it is necessary to show that

$$\sum_{m=0}^{M}F\big(q,q^m u\big)\prod_{k=0}^{m-1}G\big(q,q^k u\big)$$

converges, and

$$\prod_{m=0}^{M}G\big(q,q^m u\big)W\big(q,q^{M+1}u\big)\to 0$$

as $M\to\infty$ (both considered as power series in q and u).

Both F and G are bivariate power series in q and u. We have that

$$\begin{aligned}F(q,u)&=qu+q^2\big(u+u^2\big)+q^3\big(2u+2u^2+u^3\big)+O\big(q^4\big),\\ G(q,u)&=q(-1+u)+q^2\big(-2+u+u^2\big)+q^3\big(-4+2u+2u^2+u^3\big)+O\big(q^4\big).\end{aligned}$$

It follows that $F(q,q^mu)=O(q^{m+1})$ and $G(q,q^ku)=O(q)$ for all $m,k\geqslant 0$. So then

$$F\big(q,q^m u\big)\prod_{k=0}^{m-1}G\big(q,q^k u\big)=O\big(q^{2m+1}\big).$$

So considered as a power series in q and u, the first term in the right-hand side of (11) does converge to a fixed power series as $M\to\infty$.

By the same argument, we see that

$$\prod_{m=0}^{M}G\big(q,q^m u\big)\to 0$$

as $M \to \infty$. So it suffices to show that $W(q, q^{M+1}u)$ converges to a fixed power series. But now every term in the series $W(q,u)$ has at least one factor of u (since every polygon has positive width), so it immediately follows that $W(q, q^{M+1}u) \to 0$ as $M \to \infty$.

So both terms in (11) behave as required as $M \to \infty$, and the result follows. □

Theorem 1. *The area generating function for 3-sided prudent polygons is*

$$\begin{aligned} PA^{(3)}(q) &= \frac{-2q^3(1-q)^2}{(1-2q)^2} \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{(1-2q)^m(1-q-q^{m+1})} \prod_{k=1}^{m-1} \frac{1-q-q^k+q^{k+1}-q^{k+2}}{1-q-q^{k+1}} \\ &\quad + \frac{2q(3-10q+9q^2-q^3)}{(1-2q)^2(1-q)} \\ &= 6q + 10q^2 + 20q^3 + 42q^4 + 92q^5 + 204q^6 + 454q^7 + 1010q^8 + 2242q^9 \\ &\quad + 4962q^{10} + \cdots. \end{aligned}$$

Proof. A 3-sided prudent polygon must end at $(-1,0)$, $(0,1)$ or $(1,0)$, in either a clockwise or counter-clockwise direction. Setting $u = 1$ in $W(q,u)$ gives the area generating function

$$\begin{aligned} W(q,1) &= \frac{-q^3(1-q)^2}{(1-2q)^2} \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m}}{(1-2q)^m(1-q-q^{m+1})} \prod_{k=1}^{m-1} \frac{1-q-q^k+q^{k+1}-q^{k+2}}{1-q-q^{k+1}} \\ &\quad + \frac{q(1-q)^2}{(1-2q)^2}. \end{aligned} \tag{12}$$

A clockwise polygon ending at $(-1,0)$ can only be a single column, which has generating function

$$\frac{q}{1-q}. \tag{13}$$

A counter-clockwise polygon ending at $(0,1)$ cannot step left of the y-axis or above the line $y = 1$. While it is below this line, it must remain on the east side of its box, and upon reaching the line $y = 1$, it must step west to the y-axis. It must therefore be a bargraph, with generating function

$$\frac{q}{1-2q}. \tag{14}$$

A reflection in the y-axis converts a polygon ending at $(-1,0)$ to one ending at $(1,0)$ in the opposite direction, and reverses the direction of a polygon ending at $(0,1)$. So adding together and doubling (12), (13) and (14) will cover all possibilities, and gives the stated result. □

2.4. Enumeration of 4-sided polygons by area

This case is included for completeness, as the results are not needed in our subsequent asymptotic analysis. A 4-sided prudent polygon may end at any of $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$ in either a clockwise or counter-clockwise direction. Reflection and rotation lead to an 8-fold symmetry, so it suffices to count only those ending at $(-1,0)$ in a counter-clockwise direction. We modify Schwerdtfeger's sub-classification slightly.

Let $X(q,u,v) = \sum_{n\geqslant 1}\sum_{i\geqslant 1}\sum_{j\geqslant 1} x_{n,i,j} q^n u^i v^j$ be the generating function for those polygons (class $\mathcal{X}$) for which removing the top row does not change the width or leave two or more disconnected pieces, with q measuring area, u measuring width and v measuring height.

Let $Y(q,u,v) = \sum_{n\geqslant 1}\sum_{i\geqslant 1}\sum_{j\geqslant 1} y_{n,i,j} q^n u^i v^j$ be the generating function for the unit square plus those polygons (class $\mathcal{Y}$) not in $\mathcal{X}$ for which removing the rightmost column does not change the height or leave two or more disconnected pieces, with q measuring area, u measuring height and v measuring width.

Let $Z(q,u,v) = \sum_{n\geqslant 1}\sum_{i\geqslant 1}\sum_{j\geqslant 1} z_{n,i,j} q^n u^i v^j$ be the generating function for the polygons (class $\mathcal{Z}$) not in $\mathcal{X}$ or $\mathcal{Y}$, with q measuring area, u measuring width -1 and v measuring height. (See Fig. 4.)

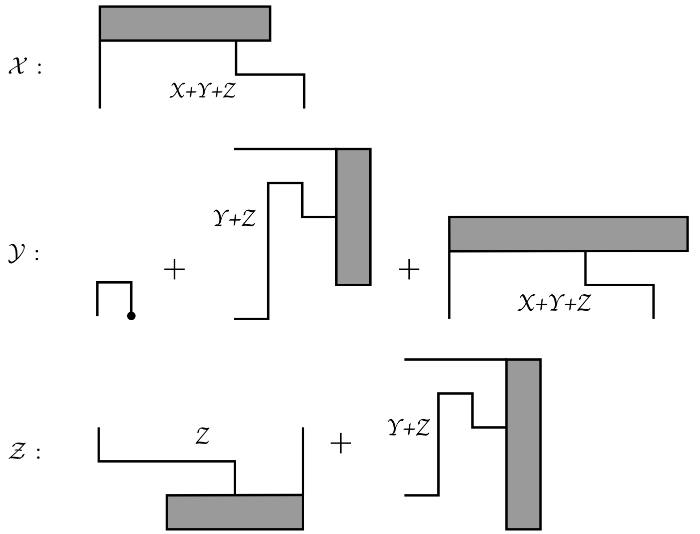

Fig. 4. The decompositions used to construct 4-sided prudent polygons in $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ (from top to bottom).

Proposition 2. *The generating functions $X(q,u,v)$, $Y(q,u,v)$ and $Z(q,u,v)$ satisfy the functional equations*

$$X(q,u,v) = \frac{qv}{1-q}\big[X(q,u,v) - X(q,qu,v)\big] + \frac{qv}{1-q}\big[Y(q,v,u) - Y(q,v,qu)\big] + \frac{quv}{1-q}\big[Z(q,u,v) - qZ(q,qu,v)\big], \tag{15}$$

$$Y(q,u,v) = quv + \frac{qv}{1-q}\big[Y(q,u,v) - Y(q,qu,v)\big] + \frac{qv^2}{1-q}\big[Z(q,v,u) - Z(q,v,qu)\big] + quv\big[X(q,qv,u) + Y(q,u,qv) + qvZ(q,qv,u)\big], \tag{16}$$

$$Z(q,u,v) = \frac{qv}{1-q}\big[Z(q,u,v) - Z(q,qu,v)\big] + qvY(q,qv,u) + quvZ(q,u,qv). \tag{17}$$

The generating function for 4*-sided prudent polygons is then given by*

$$\begin{aligned} PA^{(4)}(q) &= *8\big[X(q,1,1) + Y(q,1,1) + Z(q,1,1)\big] \\ &= 8q + 24q^2 + 80q^3 + 248q^4 + 736q^6 + 2120q^7 + 5960q^8 + 16\,464q^9 \\ &\quad + 44\,808q^{10} + \cdots. \end{aligned}$$

Proof. As with the 3-sided polygons in Lemma 1, the walk cannot visit any point (x, y) with $x, y < 0$ or with $x < -1$. The walk must approach $(-1, 0)$ from above, and must do so immediately upon reaching the line $x = -1$. So every polygon contains the north-west corner of its box. As in the 3-sided case, this leads to a construction involving adding rows to the top of existing polygons.

By definition, a polygon in $\mathcal{X}$ of width i can be constructed by adding a row of length $\leqslant i$ to the top of any polygon of width i. Adding a row to a polygon in $\mathcal{X}$ gives

$$\sum_{n\geqslant 1}\sum_{i\geqslant 1}\sum_{j\geqslant 1} x_{n,i,j} q^n u^i v^j \cdot v\sum_{k=1}^{i} q^k = qv\sum_{n\geqslant 1}\sum_{i\geqslant 1}\sum_{j\geqslant 1} x_{n,i,j} q^n u^i v^j \cdot \frac{1-q^i}{1-q},$$

which is the first term in the right-hand side of (15). Performing similar operations for polygons in $\mathcal{Y}$ and $\mathcal{Z}$ gives the rest of (15).

Note. Again, for the remainder of this subsection we give recursive relations purely in terms of the generating functions.

Polygons not in $\mathcal{X}$ must also contain the north-east corner of their box. This leads to another construction involving adding columns to the right-hand side of existing polygons. To obtain a polygon in $\mathcal{Y}$ of height i, a new column of height $\leqslant i$ should be added to a polygon of height i which contains the north-east corner of its box. So adding a column to a $\mathcal{Y}$ polygon gives

$$\frac{qv}{1-q}Y(q,u,v) - \frac{qv}{1-q}Y(q,qu,v)$$

which is the second term in the right-hand side of (16). Performing a similar operation for $\mathcal{Z}$ polygons gives the third term in (16).

Adding a new column to a polygon in $\mathcal{X}$ containing its north-east corner can be viewed as adding a sequence of rows on top of one another, and so if the new column has height $\geqslant 2$ then the resulting polygon is actually in $\mathcal{X}$. If the new column has height one, however, the resulting polygon is in $\mathcal{Y}$. Isolating those polygons in $\mathcal{X}$ which contain their north-east corner is difficult; however, we can perform an equivalent construction by adding a row of length $i+1$ to any polygon of width i. Doing so to a polygon in $\mathcal{X}$ gives

$$quvX(q,qv,u)$$

and combining this with the same for $\mathcal{Y}$ and $\mathcal{Z}$ gives the fourth term in (16). The quv term is the unit square.

Polygons in $\mathcal{Z}$ also contain the south-east corner of their box. In a similar fashion to the constructions for $\mathcal{Y}$ and $\mathcal{Z}$, we can add a new row to the bottom of a polygon containing its south-east corner. To do so to a polygon in $\mathcal{Z}$ of width $i+1$ (remember u measures width -1) requires a new row of width $\leqslant i$, so we obtain

$$\frac{qv}{1-q}Z(q,u,v) - \frac{qv}{1-q}Z(q,qu,v)$$

which is the first term in the right-hand side of (17).

Adding a new row to the bottom of something in $\mathcal{X}$ (containing its south-east corner) will give back something in $\mathcal{X}$, which will have been constructed by an alternate method described above. Adding a new row of length $\geqslant 2$ to a polygon in $\mathcal{Y}$ will result in another polygon in $\mathcal{Y}$, which will also be constructible via alternate means. So we are left only with the possibility of adding a row of length one to the bottom of a polygon in $\mathcal{Y}$. This is analogous to the above description of adding a column of height one to the right of a polygon in $\mathcal{X}$; we now proceed by adding a column of height $i+1$ to a polygon in $\mathcal{Y}$ or $\mathcal{Z}$ of height i. Doing so gives the final two terms in (17). □

3. Asymptotics

For most lattice object problems, finding and solving the functional equation(s) is the difficult part. Once a generating function has been found, the dominant singularity is often quite obvious, and so the asymptotic form of the coefficients can be easily described. The problem of 3-sided prudent polygons, however, turns out to be rather the opposite. The functional equation (7) was not terribly difficult to obtain, and its solution is relatively simple — it only comprises a sum of products of rational functions of q.

The asymptotic behavior of this model, on the other hand, is considerably more complex than any model we have seen before. The dominant singularity at $q=1/2$ is not even apparent from the representation of Theorem 1. As we shall see, there is in fact an accumulation of poles of the generating function[2] $PA(q)$ towards $q=1/2$. Accordingly, the nature of the dominant singularity at $q=1/2$ is

[2] Throughout this section only dedicated to 3-sided prudent polygons, we omit redundant superscripts and let PA_n and $PA(q)$ represent, respectively, what was denoted by $PA_n^{(3)}$ and $PA^{(3)}(q)$ in Section 2.

Quantity	At $q=1/2$	Reference
$u=\frac{q}{1-q}$	1	Eq. (23)
$v=\frac{1-q+q^2}{1-q}$	$\frac{3}{2}$	Eq. (23)
$a=\frac{q^2}{1-q+q^2}$	$\frac{1}{3}$	Eq. (41)
$\gamma=\frac{\log v}{\log 1/q}$	$\log_2(3/2)$	Eq. (49)
$C(q)=\frac{2q(3-10q+9q^2-q^3)}{(1-q)(1-2q)^2}$	$\sim\frac{1}{4(1-2q)^2}$	Eqs. (21) and (26)
$A(q)=\frac{2q(1-q)^2}{(1-2q)^2}$	$\sim\frac{1}{4(1-2q)^2}$	Eqs. (21) and (25)

Fig. 5. A table of some of the recurring quantities of Section 3, their reduction at $q=1/2$ and the relevant equations in the text.

rather unusual: a singular expansion as q approaches 1/2 can be determined, but it involves periodic fluctuations, a strong divergence from the standard simple type $Z^{\alpha}(\log Z)^{\beta}$, where $Z:=1-z/\rho$, with ρ (here equal to 1/2) the dominant singularity of the generating function under consideration. This is revealed by a Mellin analysis of $PA(q)$ near its singularity, and the periodic fluctuations, which appear to be in a logarithmic scale, eventually echo the geometric speed with which poles accumulate at 1/2. Then, thanks to a suitable extension to the complex plane, the singular expansion can be transfered to coefficients by the method known as singularity analysis [20, Chapter VI]. The next result is, for the coefficients PA_n, an asymptotic form that involves a standard element $2^n n^g$, but multiplied by a *periodic function* in $\log_2 n$. The presence of oscillations, the transcendental character of the exponent $g=\log_2 3$, and the minute amplitude of these oscillations, about 10^{-9}, are noteworthy features of this asymptotic problem.

Theorem 2. *The number $PA_n \equiv PA_n^{(3)}$ of* 3*-sided prudent polygons of area n satisfies the estimate*

$$PA_n=\left[\kappa_0+\kappa(\log_2 n)\right]2^n\cdot n^g+O\left(2^n\cdot n^{g-1}\log n\right),\quad n\to\infty, \tag{18}$$

where the critical exponent is

$$g=\log_2 3 \doteq 1.58496$$

and the "principal" constant is

$$\kappa_0=\frac{\pi}{9\log 2\sin(\pi g)\Gamma(g+1)}\prod_{j=0}^{\infty}\frac{(1-\frac{1}{3}2^{-j})(1-\frac{3}{2}2^{-j})}{(1-\frac{1}{2}2^{-j})^2}\doteq 0.1083842946. \tag{19}$$

The function $\kappa(u)$ is a smooth periodic function of u, with period 1, *mean value zero, and amplitude $\doteq 1.54623\cdot 10^{-9}$, which is determined by its Fourier series representation*:

$$\kappa(u)=\sum_{k\in\mathbb{Z}\setminus\{0\}}\kappa_k e^{2ik\pi u},\quad \textit{with } \kappa_k=\kappa_0\cdot\frac{\sin(\pi g)}{\sin(\pi g+2ik\pi^2/\log 2)}\cdot\frac{\Gamma(1+g)}{\Gamma(1+g+2ik\pi/\log 2)}.$$

The proof of the theorem occupies the next subsections, whose organization reflects the informal description given above. We shall then discuss the fine structure of subdominant terms in the asymptotic expansion of PA_n; cf. Theorem 3. Some quantities that appear repeatedly throughout this section are tabulated in Fig. 5 for convenience.

3.1. Resummations

We start with a minor reorganization of the formula provided by Theorem 1: completion of the finite products that appear there leads to the equivalent *q-hypergeometric* form

$$PA(q)=C(q)+A(q)\cdot Q(1;q)\cdot\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{(1-2q)^n}\cdot\frac{1}{Q(q^n;q)}. \tag{20}$$

Here and throughout this section, the notations are

$$C(q) := \frac{2q(3-10q+9q^2-q^3)}{(1-q)(1-2q)^2}, \qquad A(q) := \frac{2q(1-q)^2}{(1-2q)^2}, \tag{21}$$

and

$$Q(z;q) := Q\big(z;q;u(q),v(q)\big), \quad \text{where } Q(z;q;u,v) = \frac{(vz;q)_\infty}{(quz;q)_\infty}, \tag{22}$$

with

$$u(q) = \frac{q}{1-q}, \qquad v(q) = \frac{1-q+q^2}{1-q}. \tag{23}$$

In the definition of Q, the notation $(x;q)_n$ represents the usual q-Pochhammer symbol:

$$(x;q)_n = (1-x)(1-qx)\cdots\big(1-xq^{n-1}\big).$$

Lemma 3. *The function $PA(q)$ is analytic in the open disc $|q| < \sqrt{2}-1$, where it admits the convergent q-*hypergeometric *representation*

$$PA(q) = C(q) + A(q)\frac{(v;q)_\infty}{(qu;q)_\infty}\sum_{n=1}^{\infty}(-1)^n \frac{q^{2n}}{(1-2q)^n}\frac{(uq^{n+1};q)_\infty}{(vq^n;q)_\infty}, \tag{24}$$

with $A(q), C(q), u \equiv u(q), v \equiv v(q)$ rational functions given by (21) *and* (23).

Proof. The (easy) proof reduces to determining sufficient analyticity regions for the various components of the basic formula (20), some of the expansions being also of later use. First, the functions $A(q)$ and $C(q)$ are meromorphic for $|q| < 1$, with only a pole at $q = 1/2$. They can be expanded about the point $q = 1/2$ to give

$$A = \frac{1}{4(1-2q)^2} + \frac{1}{4(1-2q)} - \frac{1}{4} - \frac{1-2q}{4}, \tag{25}$$

$$C = \frac{1}{4(1-2q)^2} + \frac{5}{4(1-2q)} + \frac{3}{4} - \frac{17(1-2q)}{4} + O\big((1-2q)^2\big). \tag{26}$$

The function $Q(1;q)$ is analytic for $|q| < 1$ except at the points for which $(uq;q)_\infty = 0$, that is, the points σ for which $1 - \sigma - \sigma^n = 0$ for $n \geqslant 2$. The smallest of these (in modulus) is $\varphi = (\sqrt{5}-1)/2 = 0.618034\ldots$, a root of $1 - q - q^2$. So $Q(1;q)$ is certainly analytic at $q = 1/2$; the constant term in its expansion about $q = 1/2$ is

$$Q(1;1/2) = \frac{(3/2;1/2)_\infty}{(1/2;1/2)_\infty} = -0.18109782\ldots.$$

In similar fashion, $1/Q(z;q)$ is *bivariate analytic* at points (z,q) for which $|q| < 1$, except when $(vz;q)_\infty = 0$. This occurs at points (z_j,q) where $z_j := 1/(vq^j)$, for $j \geqslant 0$. In particular, for $|q| < \theta$, where[3]

$$\theta \doteq 0.56984 := \text{the unique real root of } 1 - 2x + x^2 - x^3, \tag{27}$$

we have $|z_0| > \theta$, hence $|z_j| > \theta$, for all $j \geqslant 0$. So, $1/Q(z;q)$ is analytic in the region $\{(z,q)\colon |z|, |q| < \theta\}$. Thus, for all $n \geqslant 1$, the functions $1/Q(q^n;q)$ are all analytic and uniformly bounded by a fixed constant, for $|q| < r_0$, where r_0 is any positive number such that $r_0 < \theta$.

[3] The function $v(q) = 1 + q^2/(1-q)$, having nonnegative Taylor coefficients, satisfies $|v(q)| \leqslant v(|q|)$, for $|q| < 1$; thus, $|1/v(q)| \geqslant 1/v(|q|)$. Also, $1/v(x)$ decreases from 1 to 0 for $x \in [0,1]$. Hence, with θ the real root of $1/v(\theta) = \theta$, it follows that $|z_0| > \theta$ as soon as $|q| < \theta$.

From these considerations, it follows that the central infinite sum that figures in (20) is, when $|q| < r_1$, dominated in modulus by a positive multiple of the series

$$\sum_n \frac{r_1^{2n}}{(1-2r_1)^n}, \tag{28}$$

provided that $r_1 < \theta$ *and* $r_1^2/(1-2r_1) < 1$. Any positive r_1 satisfying $r_1 < \sqrt{2}-1$ is then admissible. In that case, for $|q| < r_1$, the central sum is a normally convergent sum of analytic functions; hence, it is analytic. □

The radius of analyticity of $PA(q)$ is in fact 1/2. In order to obtain larger regions of analyticity, one needs to improve on the reasoning underlying the derivation of (28). This will result from a transformation of the central infinite sum in (20), namely,

$$S(q) := \sum_{n\geqslant 1} (-1)^n \frac{q^{2n}}{(1-2q)^n} \cdot \frac{1}{Q(q^n; q)}. \tag{29}$$

Only the bound $1/Q(q^n; q) = O(1)$ was used in the proof of Lemma 3, but we have, for instance, $1/Q(q^n; q) = 1 + O(q^n)$, as $n \to \infty$, and a complete expansion exists. Indeed, since $1/Q$ is bivariate analytic in $|z|, |q| < \theta$, its z-expansion at the origin is of the form

$$\frac{1}{Q(z; q)} = 1 + \sum_{\nu\geqslant 1} d_\nu(q) z^\nu. \tag{30}$$

In particular, at $z = q^n$, we have

$$\frac{1}{Q(q^n; q)} = 1 + \sum_{\nu\geqslant 1} d_\nu(q) q^{\nu n}. \tag{31}$$

Now, consider the effect of an individual term $d_\nu(q)$ (instead of $1/Q(q^n; q)$) on the sum (29). The identity

$$\sum_{n\geqslant 1} (-1)^n \frac{q^{2n}}{(1-2q)^n} q^{\nu n} = -\frac{q^{\nu+2}}{1-2q+q^{\nu+2}} \tag{32}$$

provides an analytic form for the sum on the left, as long as q is not a pole of the right-hand side. Proceeding formally, we then get, with (31) and (32), upon exchanging summations in the definition (29) of $S(q)$, a form of $PA(q)$ that involves *infinitely many meromorphic elements* of the form $1/(1-2q+q^{\nu+2})$.

We shall detail validity conditions for the resulting expansion; see (34) below. What matters, as seen from (32), is the location of poles of the rational functions $(1-2q+q^{\nu+2})^{-1}$, for $\nu \geqslant 1$. Define the quantities

$$\zeta_k := \text{the root in } [0, 1] \text{ of } 1 - 2x + x^{k+2} = 0. \tag{33}$$

We have

$$\zeta_0 = 1, \qquad \zeta_1 = \frac{\sqrt{5}-1}{2} \doteq 0.618, \qquad \zeta_2 \doteq 0.543, \qquad \zeta_3 \doteq 0.518, \qquad \ldots$$

and $\zeta_k \to \frac{1}{2}$ as k increases. The location of the complex roots of $1-2x+x^{k+2} = 0$ is discussed at length in [20, Example V.4, p. 308], as it is related to the analysis of longest runs in binary strings: a consequence of the principle of the argument (or Rouché's Theorem) is that, apart from the positive real root ζ_k, all other complex roots lie outside the disc $|z| < \frac{3}{4}$. The statement below builds upon this discussion and provides an extended analyticity region for $PA(q)$ as well as a justification of the validity of the expansion resulting from (31) and (32), which is crucial to subsequent developments.

Lemma 4. *The generating function PA(q) is analytic at all points of the slit disc*

$$\mathcal{D}_0 := \left\{ q\colon\ |q| < \frac{55}{100};\ q \notin \left[\frac{1}{2}, \frac{55}{100}\right] \right\}.$$

For $q \in \mathcal{D}_0$, the function PA(q) admits the analytic representation

$$PA(q) = C(q) - A(q)\frac{(v;q)_\infty}{(qu;q)_\infty}\left[\frac{q^2}{(1-q)^2} + \sum_{\nu\geqslant 1} d_\nu(q)\frac{q^{\nu+2}}{1-2q+q^{\nu+2}}\right], \tag{34}$$

where

$$d_\nu(q) = \left[z^\nu\right]\frac{1}{Q(z;q)} \equiv \left[z^\nu\right]\frac{(quz;q)_\infty}{(vz;q)_\infty}.$$

In the disc $|z| < \frac{55}{100}$ punctured at $\frac{1}{2}$, the function PA(q) is meromorphic with simple poles at the points $\zeta_2, \zeta_3, \ldots,$ with ζ_k as defined in (33)*. Consequently, the function PA(q) is non-holonomic, and, in particular, transcendental.*

Proof. The starting point, noted in the proof of Lemma 3, is that fact that $1/Q(z;q)$ is bivariate analytic at all points (z,q) such that $|z|, |q| < \theta$, where $\theta \doteq 0.56984$ is specified in (27). Cauchy's coefficient formula,

$$d_\nu(q) = \frac{1}{2i\pi}\int_{|z|=\theta_1} \frac{1}{Q(z;q)}\frac{dz}{z^{\nu+1}},$$

is applicable for any θ_1 such that $0 < \theta_1 < \theta$. Let us set $\theta_1 = \frac{56}{100}$. Then, since $1/Q(z;q)$ is analytic, hence continuous, hence bounded, for $|z| \leqslant \theta_1$ and $|q| \leqslant \theta_1$, trivial bounds applied to the Cauchy integral yield

$$\left|d_\nu(q)\right| < C \cdot \theta_1^{-\nu}, \tag{35}$$

for some absolute constant $C > 0$.

Consider the double sum resulting from the substitution of (31) into (29),

$$S(q) = \sum_{n\geqslant 1}(-1)^n\frac{q^{2n}}{(1-2q)^n}\cdot\left(1 + \sum_{\nu\geqslant 1} d_\nu(q)q^{\nu n}\right).$$

If we constrain q to be small, say $|q| < \frac{1}{10}$, we see from (35) that the double sum is absolutely convergent. Hence, the form (34) is justified for such small values of q. We can then proceed by analytic continuation from the right-hand side of (34). The bound (35) grants us the fact that the sum that appears there is indeed analytic in $\mathcal{D}_0$. The statements, relative to the analyticity domain and the alternative expansion (34) follow. Finally, since the value $\frac{1}{2}$ corresponds to an accumulation of poles, the function $PA(q)$ is non-holonomic (see, e.g., [16] for context). □

As an immediate consequence of the dominant singularity being at $\frac{1}{2}$, the coefficients PA_n must obey a weak asymptotic law of the form

$$PA_n = 2^n\theta(n), \quad \text{where } \limsup_{n\to\infty}\theta(n)^{1/n} = 1,$$

that is, $\theta(n)$ is a (currently unknown) subexponential factor.

More precise information requires a better characterization of the behavior of $S(q)$, as q approaches the dominant singularity $\frac{1}{2}$. This itself requires a better understanding of the coefficients $d_\nu(q)$. To this end, we state a general and easy lemma about the coefficients of quotients of q-factorials.

Lemma 5. *Let a be a fixed complex number satisfying* $|a| < 1$ *and let q satisfy* $|q - \frac{1}{2}| < \frac{1}{10}$. *One has, for* $\nu \geqslant 1$

$$[z^\nu]\frac{(az;q)_\infty}{(z;q)_\infty} = \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{(aq^{-j};q)_\infty}{(q^{-j};q)_j}\cdot q^{j\nu}. \tag{36}$$

Proof. The function $h(z) := (az;q)_\infty/(z;q)_\infty$ has simple poles at the points $\bar{z}_j := q^{-j}$, for $j \geqslant 0$. We have

$$h(z) \underset{z\to\bar{z}_j}{\sim} \frac{e_j(a;q)}{1-zq^j}, \quad e_j(a;q) := \frac{(aq^{-j};q)_\infty}{(q^{-j};q)_j(q;q)_\infty}.$$

The usual expansion of coefficients of meromorphic functions [20, Theorem IV.10, p. 258] immediately implies a terminating form for any $J \in \mathbb{Z}_{\geqslant 0}$:

$$[z^\nu]h(z) = \sum_{j=0}^{J} e_j(a;q)q^{j\nu} + O\left(R_J^n\right), \tag{37}$$

where we may adopt $R_J = \frac{3}{2}q^{-J}$.

The last estimate (37) corresponds to an evaluation by residues of the Cauchy integral representation of coefficients,

$$[z^\nu]h(z) = \frac{1}{2i\pi}\int_{|z|=R_J} h(z)\frac{dz}{z^{\nu+1}}.$$

Now, let J tend to infinity. The quantity R_J lies approximately midway between two consecutive poles, q^{-J} and q^{-J-1}, and it can be verified elementarily that, throughout $|z| = R_J$, the function $h(z)$ remains bounded in modulus by an absolute constant (this requires the condition $|a| < 1$). It then follows that we can let J tend to infinity in (37). For $\nu \geqslant 1$, the coefficient integral taken along $|z| = R_J$ tends to 0, so that, in the limit, the exact representation (36) results. □

The formula (36) is equivalent to the partial fraction expansion (Mittag–Leffler expansion; see [22, §7.10]) of the function $h(z)$, which is meromorphic in the whole complex plane:

$$\frac{(az;q)_\infty}{(z;q)_\infty} = 1 + \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{(aq^{-j};q)_\infty}{(q^{-j};q)_j}\frac{zq^j}{1-zq^j}. \tag{38}$$

(The condition $|a| < 1$ ensures the convergence of this expansion.) As observed by Christian Krattenthaler (private communication, June 2010), this last identity is itself alternatively deducible from the q-Gauß identity[4]

$${}_2\phi_1\left[\begin{matrix}A, B\\ C\end{matrix}; q, \frac{C}{AB}\right] = \frac{(C/A;q)_\infty(C/B;q)_\infty}{(C;q)_\infty(C/(AB);q)_\infty},$$

upon noticing that

$$h(z) = \frac{(a;q)_\infty}{(1-z)(q;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}q/a, z\\ qz\end{matrix}; q, a\right].$$

A direct consequence of Lemma 5 is an expression for the coefficients $d_\nu(q) = [z^\nu]Q(z;q)^{-1}$, with $Q(z;q)$ defined by (22):

$$d_\nu(q) = \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{(quv^{-1}q^{-j};q)_\infty}{(q^{-j};q)_j}\cdot\left(vq^j\right)^\nu, \quad \nu \geqslant 1. \tag{39}$$

[4] For notations, see Gasper and Rahman's reference text [21]: page 3 (definition of ${}_r\phi_s$) and Eq. (1.5.1), page 10 (q-Gauß summation).

To see this, set

$$a = quv^{-1} = \frac{q^2}{1-q+q^2},$$

and replace z by zv in the definition of $h(z)$. Note that at $q = 1/2$, we have $u = 1$, $v = 3/2$, $a = 1/3$, so that, for $q \approx 1/2$, we expect $d_\nu(q)$ to grow roughly like $(3/2)^\nu$.

Summarizing the results obtained so far, we state:

Proposition 3. *The generating function of* 3*-sided prudent polygons satisfies the identity*

$$PA(q) = D(q) - q^2 A(q) \frac{(a;q)_\infty (v;q)_\infty}{(q;q)_\infty (av;q)_\infty} \sum_{\nu=1}^{\infty} \sum_{j=0}^{\infty} \left[\frac{(aq^{-j};q)_j}{(q^{-j};q)_j} \cdot \frac{v^\nu q^{(j+1)\nu}}{1-2q+q^{\nu+2}} \right], \tag{40}$$

where

$$a = \frac{q^2}{1-q+q^2}, \qquad v = \frac{1-q+q^2}{1-q}, \qquad D(q) = C(q) - \frac{q^2}{(1-q)^2} A(q) \frac{(v;q)_\infty}{(av;q)_\infty}, \tag{41}$$

and $A(q), C(q)$ *are rational functions defined in Eq.* (21)*.*

Proof. The identity is a direct consequence of the formula (39) for $d_\nu(q)$ and of the expression for $PA(q)$ in (34), using the equivalence $av = qu$ and the simple reorganization

$$(aq^{-j};q)_\infty = (aq^{-j};q)_j \cdot (a;q)_\infty.$$

Previous developments imply that the identity (40) is, in particular, valid in the real interval $(0, \frac{1}{2})$. The trivial equality

$$\frac{(aq^{-j};q)_j}{(q^{-j};q)_j} = \frac{(a-q)(a-q^2)\cdots(a-q^j)}{(1-q)(1-q^2)\cdots(1-q^j)} \tag{42}$$

then shows that the expression on the right-hand side indeed represents a *bona fide* formal power series in q, since the q-valuation of the general term of the double sum in (40) increases with both j and ν. □

The formula (40) of Proposition 3 will serve as the starting point of the asymptotic analysis of $PA(q)$ as $q \to 1/2$ in the next subsection. Given the discussion of the analyticity of the various components in the proof of Lemma 3, the task essentially reduces to estimating the double sum in a suitable complex neighborhood of $q = 1/2$.

3.2. Mellin analysis

Let $T(q)$ be the double sum that appears in the expression (40) of $PA(q)$. We shall take it here in the form

$$T = \sum_{j=0}^{\infty} \frac{(aq^{-j};q)_j}{(q^{-j};q)_j} H_j(q) \quad \text{where } H_j(q) := \sum_{\nu=1}^{\infty} \frac{v^\nu q^{(j+1)\nu}}{1-2q+q^{\nu+2}}. \tag{43}$$

We will now study the functions H_j and propose to show that those of greater index contribute less significant terms in the asymptotic expansion of $PA(q)$ near $q = 1/2$. In this way, a complete asymptotic expansion of the function $PA(q)$, hence of its coefficients PA_n, can be obtained.

The main technique used here is that of Mellin transforms: we refer the reader to [17] for details of the method. The principles are recalled in Section 3.2.1 below. We then proceed to analyze the double sum T of (43) when q is *real* and q tends to $1/2$. The corresponding expansion is fairly explicit and it is obtained at a comparatively low computational cost in Section 3.2.2. We finally show in Section 3.2.3 that the expansion extends to a *sector of the complex plane* around $q = 1/2$.

3.2.1. Principles of the Mellin analysis

Let $f(x)$ be a complex function of the real argument x. Its *Mellin transform*, denoted by $f^\star(s)$ or $\mathcal{M}[f]$, is defined as the integral

$$\mathcal{M}[f](s) \equiv f^\star(s) := \int_0^\infty f(x)x^{s-1}\,dx, \tag{44}$$

where s may be complex. It is assumed that $f(x)$ is locally integrable. It is then well known that if f satisfies the two asymptotic conditions

$$f(x) \underset{x\to 0}{=} O\bigl(x^\alpha\bigr), \qquad f(x) \underset{x\to +\infty}{=} O\bigl(x^\beta\bigr),$$

with $\alpha > \beta$, then $f^\star$ is an analytic function of s in the strip of the complex plane,

$$-\alpha < \Re(s) < -\beta,$$

also known as a *fundamental strip*. Then, with c any real number of the interval $(-\alpha, -\beta)$, the following inversion formula holds (see [38, §VI.9] for detailed statements):

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} f^\star(s)x^{-s}\,ds. \tag{45}$$

There are then two essential properties of Mellin transforms.

(M_1) *Harmonic sum property.* If the pairs (λ, μ) range over a denumerable subset of $\mathbb{R} \times \mathbb{R}_{>0}$ then one has the equality

$$\mathcal{M}\Biggl[\sum_{(\lambda,\mu)} \lambda f(\mu x)\Biggr] = f^\star(s)\cdot\Biggl(\sum_{(\lambda,\mu)} \lambda\mu^{-s}\Biggr). \tag{46}$$

That is to say, the harmonic sum $\sum \lambda f(\mu x)$ has a Mellin transform that decomposes as a product involving the transform of the base function ($f^\star$) and the generalized Dirichlet series ($\sum \lambda\mu^{-s}$) associated with the "amplitudes" λ and the "frequencies" μ. Detailed validity conditions, spelled out in [17], are that the exchange of summation ($\sum$, in the definition of the harmonic sum) and integral ($\int$, in the definition of the Mellin transform) be permissible.

(M_2) *Mapping properties.* Poles of transforms are in correspondence with asymptotic expansions of the original function. More precisely, if the Mellin transform $F^\star$ of a function F admits a meromorphic extension beyond the fundamental strip, with a pole of some order m at some point $s_0 \in \mathbb{C}$, with $\Re(s_0) < -\alpha$, then it contributes an asymptotic term of the form $P(\log x)x^{-s_0}$ in the expansion of $F(x)$ as $x \to 0$, where P is a computable polynomial of degree $m-1$. Schematically:

$$F^\star(s) \underset{s\to s_0}{:} \frac{C}{(s-s_0)^m} \quad\Longrightarrow\quad F(x) \underset{x\to 0}{:} P(\log x)x^{-s_0} = \operatorname{Res}\bigl(f^\star(s)x^{-s}\bigr)_{s=s_0}. \tag{47}$$

Detailed validity conditions, again spelled out in [17], are a suitable decay of the transform $F^\star(s)$, as $\Im(s) \to \pm\infty$, so as to permit an estimate of the inverse Mellin integral (45) by residues — in (47), the expression is then none other than the residue of $f^\star(s)x^{-s}$ at $s = s_0$.

The power of the Mellin transform for the asymptotic analysis of sums devolves from the application of the mapping property (M_2) to functions $F(x) = \sum \lambda f(\mu x)$ that are harmonic sums in the sense of (M_1). Indeed, the factorization property (46) of (M_1) makes it possible to analyze separately the singularities that arise from the base function (via $f^\star$) and from the amplitude–frequency pairs (via $\sum \lambda\mu^{-s}$); hence an asymptotic analysis results, thanks to (M_2).

3.2.2. Analysis for real values of $q \to 1/2$

Our purpose now is to analyze the quantity T of (43) with $q < 1/2$, when $q \to 1/2$. This basically reduces to analyzing the quantities $H_j(q)$ of (43). Our approach consists of setting $t = 1 - 2q$ and *decoupling*[5] the quantities t and q. Accordingly, we define the function

$$h_j(t) \equiv h_j(t; q, \nu) := q^{-2} \sum_{\nu=1}^{\infty} \frac{(\nu q^j)^\nu}{1 + tq^{-\nu-2}}, \tag{48}$$

so that

$$H_j(q) = h_j\big(t; q, \nu(q)\big),$$

with the definition (43). We shall let t range over $\mathbb{R}_{\geqslant 0}$ but restrict the parameter q to a small interval $(1/2 - \epsilon_0, 1/2 + \epsilon_0)$ of $\mathbb{R}$ and the parameter ν to a small interval of the form $(3/2 - \epsilon_1, 3/2 + \epsilon_1)$, since $\nu(1/2) = 3/2$. We shall write such a restriction as

$$q \approx \frac{1}{2}, \qquad \nu \approx \frac{3}{2},$$

with the understanding that ϵ_0, ϵ_1 can be taken suitably small, as the need arises. Thus, *for the time being, we ignore the relations that exist between t and the pair q, ν, and we shall consider them as independent quantities*.

As a preamble to the Mellin analysis, we state an elementary lemma.

Lemma 6. *Let q be restricted to a sufficiently small interval containing $1/2$ and ν to a sufficiently small interval containing $3/2$. Each function $h_j(t)$ defined by (48) satisfies the estimate*

$$h_j(t) \underset{t\to+\infty}{=} O\left(\frac{1}{t}\right), \qquad h_j(t) \underset{t\to 0}{=} \begin{cases} O(1) & \text{if } j \geqslant 1, \\ O(t^{-\gamma}) & \text{if } j = 0, \text{ with } \gamma = \frac{\log \nu}{\log(1/q)}. \end{cases} \tag{49}$$

For γ, we can also adopt any fixed value larger than $\log_2(4/3) \doteq 0.415$, provided q and ν are taken close enough to $1/2$ and $3/2$, respectively.

Proof. *Behavior as $t \to +\infty$.* The inequality $(1 + tq^{-\nu-2})^{-1} < t^{-1}q^{\nu+2}$ implies by summation the inequality

$$h_j(t) \leqslant q^{-2}t^{-1} \sum_{\nu=1}^{\infty} \nu^\nu q^{j\nu} q^\nu = O\left(\frac{1}{t}\right), \quad t \to +\infty,$$

given the convergence of the geometric series $\sum_\nu \nu q^{(j+1)\nu}$, for $\nu \approx 3/2$ and $q \approx 1/2$.

Behavior as $t \to 0$. First, for the easy case $j \geqslant 1$, the trivial inequality $(1 + tq^{-\nu-2})^{-1} \leqslant 1$ implies

$$h_j(t) = O\left(\sum_\nu \big(\nu q^j\big)^\nu\right) = O(1), \quad t \to 0.$$

Next, for $j = 0$, define the function

$$\nu_0(t) := -2 + \frac{\log(1/t)}{\log(1/q)},$$

so that $tq^{-\nu-2} < 1$, if $\nu < \nu_0(t)$, and $tq^{-\nu-2} \geqslant 1$, if $\nu \geqslant \nu_0(t)$. Write $\sum_\nu = \sum_{\nu_0} + \sum_{\nu \geqslant \nu_0}$. The sum corresponding to $\nu \geqslant \nu_0$ is bounded from above as in the case of $t \to +\infty$,

$$\sum_{\nu \geqslant \nu_0(t)} \frac{\nu^\nu}{1 + tq^{-\nu-2}} \leqslant \sum_{\nu \geqslant \nu_0(t)} \nu^\nu t^{-1} q^{\nu+2} = O\big(t^{-1}(\nu q)^{\nu_0}\big) = O\big(t^{-1}(\nu q)^{\nu_0}\big), \quad t \to 0,$$

[5] An instance of such a decoupling technique appears for instance in de Bruijn's reference text [5, p. 27].

and the last quantity is $O(t^{-\gamma})$ for $\gamma = (\log v)/\log(1/q)$. The sum corresponding to $\nu < \nu_0$ is dominated by its later terms and is accordingly found to be $O(t^{-\gamma})$. The estimate of $h_0(t)$, as $t \to 0$, results. □

We can now proceed with a precise asymptotic analysis of the functions $h_j(t)$, as $t \to 0$. Lemma 6 implies that each $h_j(t)$ has its Mellin transform $h_j^\star(s)$ that exists in a non-empty fundamental strip left of $\Re(s) = 1$. In that strip, the Mellin transform is

$$\begin{aligned}\mathcal{M}\big[h_j(t)\big] &= q^{-2}\mathcal{M}\left[\frac{1}{1+t}\right] \cdot \left(\sum_{\nu=1}^{\infty} \big(vq^j\big)^{\nu} \big(q^{-\nu-2}\big)^{-s}\right) \\ &\qquad \big(\text{by the harmonic sum property } (\mathrm{M}_1)\big) \\ &= q^{-2}\mathcal{M}\left[\frac{1}{1+t}\right] \cdot \frac{vq^{j+3s}}{1-vq^{j+s}} \quad (\text{by summation of a geometric progression}) \\ &= q^{-2}\frac{\pi}{\sin \pi s}\frac{vq^{j+3s}}{1-vq^{j+s}} \quad \big(\text{by the classical form of } \mathcal{M}\big[(1+t)^{-1}\big]\big). \end{aligned} \tag{50}$$

The Mellin transform of $(1+t)$, which equals $\pi/\sin(\pi s)$, admits $0 < \Re(s) < 1$ as the fundamental strip, so this condition is necessary for the validity of (50). In addition, the summability of the Dirichlet series, here plainly a geometric series, requires the condition $|vq^{j+s}| < 1$; that is,

$$\Re(s) > -j + \frac{\log v}{\log 1/q}.$$

In summary, the validity of (50) is ensured for s satisfying

$$\lambda < \Re(s) < 1, \quad \text{with } \lambda := \max\left(0, -j + \frac{\log v}{\log 1/q}\right).$$

Lemma 7. *For $q \approx 1/2$ and $v \approx 3/2$ restricted as in Lemma* 6, *the function $h_j(t)$ admits an* exact representation, *valid for any $t \in (0, q^{-3})$,*

$$h_j(t) = (-1)^j \frac{vq^{3\gamma-2j-2}}{\log 1/q} t^{j-\gamma} \Pi(\log_{1/q} t) + q^{-2} \sum_{r \geqslant 0} (-1)^r \frac{vq^{j-3r}}{1-vq^{j-r}} t^r. \tag{51}$$

Here,

$$\gamma \equiv \gamma(q) := \frac{\log v}{\log 1/q}$$

so that $\gamma \approx \log_2 \frac{3}{2} \doteq 0.415$, when $q \approx \frac{1}{2}$; the quantity $\Pi(u)$ is an absolutely convergent Fourier series,

$$\Pi(u) := \sum_{k \in \mathbb{Z}} p_k e^{-2ik\pi u}, \tag{52}$$

with coefficients p_k given explicitly by

$$p_k = \frac{\pi}{\sin(\pi\gamma + 2ik\pi^2/(\log 1/q))}. \tag{53}$$

Observe that the p_k decrease geometrically with k. For instance, at $q = 1/2$, one has

$$p_k = O\big(e^{-2k\pi^2/\log 2}\big) \doteq O\big(4.28 \cdot 10^{-13}\big)^k, \tag{54}$$

as is apparent from the exponential form of the sine function. Consequently, even the very first coefficients are small: at $q = 1/2$, typically,

$$|p_1| = |p_{-1}| \doteq 2.69 \cdot 10^{-12}, \qquad |p_2| = |p_{-2}| \doteq 1.15 \cdot 10^{-24}, \qquad |p_3| = |p_{-3}| \doteq 4.95 \cdot 10^{-37}.$$

Proof of Lemma 7. We first perform an *asymptotic* analysis of $h_j(t)$ as $t \to 0^+$. This requires the determination of poles to the left of the fundamental strip of $h_j^\star(s)$, and these arise from two sources.

- The relevant poles of $\pi/\sin\pi s$ are at $s = 0, -1, -2, \ldots$; they are simple and the residue at $s = -r$ is $(-1)^r$.
- The quantity $(1 - vq^{j+s})^{-1}$ has a simple pole at the real point

$$\sigma_0 := -j + \frac{\log v}{\log 1/q}, \tag{55}$$

as well as *complex poles* of real part σ_0, due to the complex periodicity of the exponential function ($e^{t+2i\pi} = e^t$). The set of all poles of $(1 - vq^{j+s})^{-1}$ is then

$$\left\{\sigma_0 + \frac{2ik\pi}{\log 1/q},\ k \in \mathbb{Z}\right\}.$$

The proof of an asymptotic representation (that is, of (51), with '$\sim$' replacing the equality sign there) is classically obtained by integrating $h_j^\star(s)t^{-s}$ along a long rectangle with corners at $-d - iT$ and $c + iT$, where c lies within the fundamental strip (in particular, between 0 and 1) and d will be taken to be of the form $-m - \frac{1}{2}$, with $m \in \mathbb{Z}_{\geqslant 0}$, and smaller than $-j + \gamma$. In the case considered here, there are regularly spaced poles along $\Re(s) = -j + \gamma$, so that one should take values of T that are such that the line $\Im(s) = T$ passes half-way between poles. This, given the fast decay of $\pi/\sin\pi s$ as $|\Im(s)|$ increases and the boundedness of the Dirichlet series $(1 - vq^{j+s})^{-1}$ along $\Im(s) = \pm T$, allows us to let T tend to infinity. By the Residue Theorem applied to the inverse Mellin integral (45), we collect in this way the contribution of *all* the poles at $-j + \gamma + 2ik\pi/(\log 1/q)$, with $k \in \mathbb{Z}$, as well as the $m + 1$ initial terms of the sum $\sum_r$ in (51). The resulting expansion is of type (51) with the sum $\sum_r$ truncated to $m + 1$ terms and an error term that is $O(t^{m+1/2})$.

In general, what the Mellin transform method gives is an *asymptotic* rather than *exact* representation of this type. Here, we have more. We can finally let m tend to infinity and verify that the inverse Mellin integral (45) taken along the vertical line $\Re(s) = -m - \frac{1}{2}$ remains uniformly bounded in modulus by a quantity of the form $ct^m q^{-3m}$, for some $c > 0$. In the limit $m \to +\infty$, the integral vanishes (as long as $tq^{-3} < 1$), and the exact representation (51) is obtained. □

We can now combine the *identity* provided by Lemma 7 with the decomposition of the generating function $PA(q)$ as allowed by Eq. (43), which flows from Proposition 3. We recall that $H_j(q) = h_j(t; q, v(q))$.

Proposition 4. *The generating function $PA(q)$ of prudent polygons satisfies, for q in a small enough interval*[6] *of the form $(1/2 - \epsilon, 1/2)$ (for some $\epsilon > 0$), the* identity

$$PA(q) = D(q) - q^2 A(q)\frac{(a;q)_\infty (v;q)_\infty}{(q;q)_\infty (av;q)_\infty} T(q), \tag{56}$$

where the notations are those of Proposition 3, and the function $T(q)$ admits the exact representation

$$T(q) = (1-2q)^{-\gamma} \cdot \Pi\left(\frac{\log(1-2q)}{\log 1/q}\right) U(q) + V(q), \quad \gamma \equiv \frac{\log v}{\log 1/q}, \tag{57}$$

with $\Pi(u)$ given by Lemma 7, Eqs. (52) and (53). Set

$$t = 1 - 2q.$$

The "singular series" $U(q)$ *is*

$$U(q) = \frac{vq^{3\gamma-2}}{\log 1/q}\frac{(-q^{-1}t;q)_\infty}{(-aq^{-2}t;q)_\infty}, \quad \gamma = \frac{\log v}{\log 1/q}; \tag{58}$$

[6] Numerical experiments suggest that in fact the formula (57) remains valid for all $q \in (0, 1/2)$.

and the "regular series" $V(q)$ *is*

$$V(q) = -\frac{(q;q)_\infty}{(a;q)_\infty}\frac{q^{-2}}{1+q^{-2}t} + q^{-2}\frac{(q;q)_\infty(av;q)_\infty}{(a;q)_\infty(v;q)_\infty}\sum_{r=0}^{\infty}\frac{(a^{-1}v^{-1}q;q)_r}{(v^{-1}q;q)_r}\left(-aq^{-2}t\right)^r. \tag{59}$$

Proof. We start from $T(q)$ as defined by (43). The q-binomial theorem is the identity [21, §1.3]

$$\frac{(\theta z;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(\theta;q)_n}{(q;q)_n}z^n. \tag{60}$$

Now consider the first term in the expansion (51) of Lemma 7. Sum the corresponding contributions for all values of $j \geqslant 0$, after multiplication by the coefficient $(aq^{-j};q)_j/(q^{-j};q)_j$, in accordance with (43). This gives

$$U(q) = \frac{vq^{3\gamma-2}}{\log 1/q}\sum_{j=0}^{\infty}\frac{(aq^{-j};q)_j}{(q^{-j};q)_j}\left(-q^2t\right)^j = \frac{vq^{3\gamma-2}}{\log 1/q}\sum_{j=0}^{\infty}\frac{(a^{-1}q;q)_j}{(q;q)_j}\left(-aq^{-2}t\right)^j$$

which provides the expression for $U(q)$ of the singular series, via the q-binomial theorem (60) taken with $z = -at$ and $\theta = a^{-1}q$.

Summing over j in the second term in the identity (51) of Lemma 7, we have

$$V(q) = q^{-2}\sum_{r=0}^{\infty}\left(-q^{-2}t\right)^r\sum_{j=0}^{\infty}\frac{(aq^{-j};q)_j}{(q^{-j};q)_j}\frac{vq^{j-r}}{1-vq^{j-r}}.$$

Now, the Mittag–Leffler expansion (38) associated with Lemma 5 can be put in the form

$$\frac{(az;q)_\infty}{(z;q)_\infty} = 1 + \frac{(a;q)_\infty}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{(aq^{-j};q)_j}{(q^{-j};q)_j}\frac{zq^j}{1-zq^j}.$$

An application of this identity to $V(q)$, with $z = vq^{-r}$, shows that

$$V(q) = q^{-2}\frac{(q;q)_\infty}{(a;q)_\infty}\sum_{r=0}^{\infty}\left(-q^{-2}t\right)^r\left(\frac{(avq^{-r};q)_\infty}{(vq^{-r};q)_\infty} - 1\right),$$

which is equivalent to the stated form of $V(q)$. Note that this last form is a q-hypergeometric function of type ${}_2\phi_1$; see [21].

So far, we have proceeded formally and left aside considerations of convergence. It can be easily verified that all the sums, single or double, involved in the calculations above are absolutely (and uniformly) convergent, provided t is taken small enough (i.e., q is sufficiently close to $1/2$), given that all the involved parameters, such as a, u, v, then stay in suitably bounded intervals of the real line. □

3.2.3. *Analysis for complex values of* $q \to 1/2$

We now propose to show that the "transcendental" expression of $PA(q)$ provided by Proposition 4 is actually valid in certain regions of the complex plane that extend beyond an interval of the real line. The regions to be considered are dictated by the requirements of the singularity analysis method to be deployed in the next subsection.

Definition 3. *Let* θ_0 *be a number in the interval* $(0, \pi/2)$, *called the* angle, *and* r_0 *a number in* $\mathbb{R}_{>0}$, *called the* radius. *A* sector (*anchored at* $1/2$) *is comprised of the set of all complex numbers* $z = 1/2 + re^{i\theta}$ *such that*

$$0 < r < r_0 \quad \text{and} \quad \theta_0 < \theta < 2\pi - \theta_0.$$

We stress the fact that the angle should be strictly smaller than $\pi/2$, so that a sector in the sense of the definition always includes a part of the line $\Re(s) = 1/2$. The smallness of a sector will be measured by the smallness of r_0. That is to say:

Proposition 5. *There exists a sector $\mathcal{S}_0$ (anchored at 1/2), of angle[7] $\theta_0 < \pi/2$ and radius $r_0 > 0$, such that the identity expressed by Eqs.* (56) *and* (57) *holds for all $q \in \mathcal{S}_0$.*

Proof. The proof is a simple consequence of analytic continuation. We first observe that an infinite product such as $(c;q)_\infty$ is an analytic function of both c and q, for arbitrary c and $|q| < 1$. Similarly, the inverse $1/(c;q)_\infty$ is analytic provided $cq^j \neq 1$, for all c. For instance, taking $c = a$ where $a = a(q) = q^2/(1-q+q^2)$ and noting that $a(1/2) = 1/3$, we see that $1/(a;q)$ is an analytic function of q in a small complex neighborhood of $q = 1/2$. This reasoning can be applied to the various Pochhammer symbols that appear in the definition of $T(q)$, $U(q)$, $V(q)$. Similarly, the hypergeometric sum that appears in the regular series $V(q)$ is seen to be analytic in the three quantities $a \approx 1/3$, $v \approx 3/2$, and $t = 1-2q \approx 0$. In particular, *the functions $U(q)$ and $V(q)$ are analytic in a complex neighborhood of $q = 1/2$.*

Next, consider the quantity

$$(1-2q)^{-\gamma} = \exp\bigl(-\gamma \log(1-2q)\bigr).$$

The function $\gamma \equiv \gamma(q)$ is analytic in a neighborhood of $q = 1/2$, since it equals $(\log v)/(\log 1/q)$. The logarithm, $\log(1-2q)$, is analytic in any *sector* anchored at 1/2. By composition, there results that $(1-2q)^{-\gamma}$ is analytic in a small sector anchored at 1/2. It only remains to consider the Π factor in (56). A single Fourier element, $p_k e^{-2ik\pi u}$, with $u = \log_{1/q} t$ and $t = 1-2q$, is also analytic in a small sector (anchored at 1/2), as can be seen from the expression

$$p_k e^{-2ik\pi u} = p_k \exp\left(-2ik\pi \frac{\log(1-2q)}{\log 1/q}\right). \tag{61}$$

Note that, although $\Re(\log(1-2q)) \to \infty$ as $q \to 1/2$, the complex exponential $\exp(2ik\pi \log_2(1-2q))$ remains uniformly bounded, since $\Im(\log(1-2q))$ is bounded for q in a sector. Then, given the fast geometric decay of the coefficients p_k at $q = 1/2$ (namely, $p_k = O(e^{-2k\pi^2/\log 2})$; cf. (53)), it follows that $\Pi(\log_2 t)$ is also analytic in a sector. A crude adjustment of this argument (see (70) and (71) below for related expansions) suffices to verify that the geometric decay of the terms composing (61) persists in a sector anchored at 1/2, so that $\Pi(\log_{1/q} t)$ is also analytic in such a sector.

Finally, the auxiliary quantities $D(q)$, $A(q)$ are meromorphic at $q = 1/2$, with at most a double pole there; in particular, they are analytic in a small enough sector anchored at 1/2. We can then choose for $\mathcal{S}_0$ a small sector that satisfies this as well as all the previous analyticity constraints. Then, by *unicity of analytic continuation*, the expression on the right-hand side of (56), with $T(q)$ as given by (57), *must* coincide with (the analytic continuation of) $PA(q)$ in the sector $\mathcal{S}_0$. □

3.3. Singularity analysis and transfer

If we drastically reduce all the non-singular quantities that occur in the main form (56) of Proposition 4 by letting $q \to 1/2$, we are led to infer that $PA(q)$ satisfies, in a sector around $q = 1/2$, an estimate of the form

$$PA(q) = \xi_0 (1-2q)^{-\gamma_0 - 2} \Pi\bigl(\log_2(1-2q)\bigr) + O\bigl((1-2q)^{-3/2}\bigr), \quad \gamma_0 := \log_2(3/2), \tag{62}$$

where

$$\xi_0 = -\frac{1}{16} U(1/2) \frac{(1/3;1/2)_\infty (3/2;1/2)_\infty}{(1/2;1/2)_\infty (1/2;1/2)_\infty}, \quad U(1/2) = \frac{16}{9\log 2}, \tag{63}$$

[7] A careful examination of the proof of Proposition 5 shows that *any* angle $\theta_0 > 0$, however small, is suitable, but only the existence of *some* $\theta_0 < \pi/2$ is needed for singularity analysis.

and $U(q)$ is the singular series of (58). Let us ignore for the moment the oscillating terms and simplify $\Pi(u)$ to its constant term p_0, with p_k given by (53). This provides a numerical approximation $\widehat{PA}(q)$ of $PA(q)$. With the general asymptotic approximation (derived from Stirling's formula)

$$\left[q^n\right](1-2q)^{-\lambda} \underset{n\to+\infty}{\sim} \frac{1}{\Gamma(\lambda)} 2^n n^{\lambda-1}, \quad \lambda \notin \mathbb{Z}_{\leqslant 0}, \tag{64}$$

it is easily seen that $[q^n]\widehat{PA}(q)$ is asymptotic to the quantity $\kappa_0 2^n n^g$ of Eq. (18) in Theorem 2, which is indeed the "principal" asymptotic term of $PA_n = [q^n]PA(q)$, where $g = \gamma_0 + 1 = \log_2 3$.

A rigorous justification and a complete analysis depend on the general singularity analysis theory [20, Chapter VI] applied to the expansion of $PA(q)$ near $q = 1/2$. We recall that a Δ-*domain with base* 1 is defined to be the intersection of a disc of radius strictly larger than 1 and of the complement of a sector of the form $-\theta_0 < \arg(z-1) < \theta_0$ for some $\theta_0 \in (0, \pi/2)$. A Δ-domain with base ρ is obtained from a Δ-domain with base 1 by means of the homothetic transformation $z \mapsto \rho z$. Singularity analysis theory is then based on two types of results.

(S_1) Coefficients of functions in a basic asymptotic scale have known asymptotic expansions [20, Theorem VI.1, p. 381]. In the case of the scale $(1-z)^{-\lambda}$, the expansion, which extends (64), is of the form

$$\left[z^n\right](1-z)^{-\lambda} \underset{n\to+\infty}{\sim} n^{\lambda-1}\left(1 + \sum_{k\geqslant 1} \frac{e_k}{n^k}\right), \quad \lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geqslant 0},$$

where e_k is a computable polynomial in λ of degree $2k$. Observe that this expansion is valid for *complex* values of the exponent λ, and if $\lambda = \sigma + i\tau$, then

$$n^{\lambda-1} = n^{\sigma-1} \cdot n^{i\tau} = n^{\sigma-1} e^{i\tau \log n}.$$

Thus, the real part (σ) of the singular exponent drives the asymptotic regime; the imaginary part, as soon as it is non-zero, induces *periodic oscillations* in the scale of $\log n$. A noteworthy feature is that smaller functions at the singularity $z = 1$ have asymptotically smaller coefficients.

(S_2) An approximation of a function near its singularity can be transferred to an approximation of coefficients according to the rule

$$f(z) \underset{z\to 1}{=} O\left((1-z)^{-\lambda}\right) \quad \Longrightarrow \quad \left[z^n\right] f(z) \underset{n\to+\infty}{=} O\left(n^{\lambda-1}\right).$$

The condition is that $f(z)$ be analytic in a Δ-domain and that the O-approximation holds in such a Δ-domain, as $z \to 1$; see [20, Theorem VI.3, p. 390]. Once more, smaller error terms are associated with smaller coefficients.

Equipped with these principles, it is possible to obtain a complete asymptotic expansion of $[q^n]PA(q)$ once a complete expansion of $PA(q)$ in the vicinity of $q = 1/2$ has been obtained (set $q = z/2$, so that $z \approx 1$ corresponds to $q = 1/2$). In this context, Proposition 3 precisely grants us the analytic continuation of $PA(q)$ in a Δ-domain anchored at 1/2, with any opening angle arbitrarily small; Proposition 4, together with Proposition 5, describes in a precise manner the asymptotic form of $PA(q)$ as $q \to 1/2$ in a Δ-domain and it is a formal exercise to transform them into a standard asymptotic expansion, in the form required by singularity analysis theory.

Proposition 6. *As $q \to 1/2$ in a Δ-domain, the function $PA(q)$ satisfies the expansion*

$$PA(q) \underset{q\to 1/2}{\sim} \frac{1}{(1-2q)} + R(q) + \sum_{j\geqslant 1} (1-2q)^{-\gamma_0-2+j} \sum_{\ell=0}^{j} \left(\log(1-2q)\right)^{\ell} \Pi^{(j,\ell)}\left(\log_2(1-2q)\right),$$

$$\gamma_0 = \log_2 \frac{3}{2}. \tag{65}$$

Here $R(q)$ is analytic at $q = 1/2$ and each $\Pi^{(j,\ell)}(u)$ is a Fourier series

$$\Pi^{(j,\ell)}(u) := \sum_{k\in\mathbb{Z}} p_k^{(j,\ell)} e^{2ik\pi u},$$

with a computable sequence of coefficients $p_k^{(j,\ell)}$.

Proof. From Proposition 4, we have

$$PA(q) = PA^{\text{reg}}(q) + PA^{\text{sing}}(q), \tag{66}$$

where the two terms correspond, respectively, to the "regular" part (involving the regular series $V(q)$) and the "singular part" (involving the singular series $U(q)$ as well as the factor $(1-2q)^{-\gamma}$ and the oscillating series).

Regarding the regular part, we have, with the notations of Proposition 4,

$$PA^{\text{reg}}(q) = D(q) - q^2 A(q) \frac{(a;q)_\infty (v;q)_\infty}{(q;q)_\infty (q;q)_\infty} V(q). \tag{67}$$

We already know that $A(q)$ and $D(q)$ are meromorphic at $q = 2$ with a double pole, while $V(q)$ and the Pochhammer symbols are analytic at $q = 1/2$. Thus, this regular part has at most a double pole at $q = 1/2$. A simple computation shows that the coefficient of $(1-2q)^{-2}$ reduces algebraically trivially — in the sense that no q-identity is involved — to 0. Thus, the regular part involves only a simple pole at $q = 1/2$, as is expressed by the first two terms of the expansion (66), where $R(q)$ is analytic at $q = 1/2$. (Note that the coefficient of $(1-2q)^{-1}$ is exactly 1, again for trivial reasons.)

The singular part is more interesting and it can be analyzed by the method suggested at the beginning of this subsection. Whenever convenient, we freely use the abbreviation $t = 1 - 2q$. The function $\gamma(q) = (\log v)/(\log 1/q)$ is analytic at $q = 1/2$, where

$$\begin{aligned}\gamma(q) &= \log_2 \frac{3}{2} + 2\frac{\log 3}{(\log 2)^2}\left(q - \frac{1}{2}\right) + \cdots \\ &\doteq 0.58496 + 4.5732\left(q-\frac{1}{2}\right) + 16.317\left(q-\frac{1}{2}\right)^2 + 39.982\left(q-\frac{1}{2}\right)^3 \\ &\quad + 86.991\left(q-\frac{1}{2}\right)^4 + \cdots.\end{aligned}$$

The function $(1-2q)^{-\gamma}$ can then be expanded as

$$\begin{aligned}(1-2q)^{-\gamma(q)} &= (1-2q)^{-\gamma_0} e^{-(\gamma(q)-\gamma_0)\log t}, \quad \text{with } \gamma_0 = \gamma(1/2) = \log_2 \frac{3}{2} \\ &= (1-2q)^{-\gamma_0}\left(1 + \frac{\log 3}{(\log 2)^2} t\log t + t^2 P_2(\log t) + t^3 P_3(\log t) + \cdots\right),\end{aligned} \tag{68}$$

for a computable family of polynomials $P_2, P_3, \ldots,$ where $\deg P_\ell = \ell$ and $P_\ell(0) = 0$. For instance, we have, with $y := \log t$:

$$\begin{aligned}(1-2q)^{-(\gamma-\gamma_0)\log t} &\doteq 1 + 2.28ty + t^2\left(-4.07y + 2.61y^2\right) + t^3\left(4.99y - 9.32y^2 + 1.99y^3\right) \\ &\quad + \cdots.\end{aligned}$$

The singular series $U(q)$ of (58) is analytic at $q = 1/2$ and its coefficients can be determined, both numerically and, in principle, symbolically in terms of Pochhammer symbols and their logarithmic derivatives (which lead to q-analogues of harmonic numbers). Numerically, they can be estimated to high precision, by bounding the infinite sum and products to a finite but large value. (The validity of the process can be checked empirically by increasing the values of this threshold, the justification being that all involved sums and products converge geometrically fast — we found that replacing $+\infty$

by 100 in numerical computations provides estimates that are at least correct to 25 decimal digits.) In this way, we obtain, for instance, the expansion of the function $V(q)$, which is of the form ($t = 1 - 2q$)

$$U(q) \doteq \frac{16}{9\log 2} + 9.97t + 21.5t^2 + 35.8t^3 + 51.9t^4 + \cdots. \tag{69}$$

Finally, regarding $\Pi(u)$ taken at $u = \log_{1/q}(1-2q)$, we note that the coefficients p_k of (53) can be expanded around $q = 1/2$ and pose no difficulty, while the quantities $e^{2ik\pi u}$ can be expanded by a process analogous to (68). Indeed, we have

$$p_k \equiv p_k(q) = \frac{\pi}{\sin(\pi\gamma_0 + 2ik\pi^2/\log 2)} \cdot \exp\bigl(1 + e_1(k)t + e_2(k)t^2 + \cdots\bigr), \tag{70}$$

where the e_k only grow polynomially with k. Also, at $u = \log_{1/q}(1-2q)$, one has

$$e^{2ik\pi u} = (1-2q)^{2ik\pi/\log 2} \exp\bigl[2ik\pi \log_2 t\bigl(g_1 t + g_2 t^2 + \cdots\bigr)\bigr], \tag{71}$$

where the coefficients g_j are those of $(\log 1/q)^{-1} - (\log 2)^{-1}$ expanded at $q = 1/2$ and expressed in terms of $t = 1 - 2q$.

We can now recapitulate the results of the discussion of the singular part: from (69), (70), and (71), we find that the terms appearing in the singular expansion of $PA(q)$ are of the form, for $j = 0, 1, 2, \ldots,$

$$(1-2q)^{-\gamma_0 - 2} t^j (\log t)^\ell t^{2ik\pi/\log 2},$$

with ℓ such that $0 \leqslant \ell \leqslant j$ and $k \in \mathbb{Z}$. The terms at fixed j, ℓ add up to form the Fourier series $\Pi^{(j,\ell)}$, whose coefficients exhibit a fast decrease with $|k|$, similar to that encountered in (54). Consequently, a finite version of (65) at any order holds, so that the statement results. □

With this last proposition, we can conclude the proof of Theorem 2.

Proof of Theorem 2. The analytic term $R(q)$ in (65) leaves no trace in the asymptotic form of coefficients. Thus the global contribution of the *regular* part to coefficients PA_n reduces to 2^n (with coefficient 1 and no power of n), corresponding to the term $(1-2q)^{-1}$ in (65).

The transfer to coefficients of each term of the *singular* part of (65) is permissible, given the principles of singularity analysis recalled above. Only an amended form allowing for logarithmic factors is needed, but this is covered by the general theory: for the translation of the coefficients of the basic scale $(1-z)^{-\lambda} \log^k(1-z)$, see [20, p. 387]. From a computational point of view, one may conveniently operate [20, Note VI.7, p. 389] with

$$\bigl[z^n\bigr](1-z)^{-\lambda}\bigl(\log(1-z)\bigr)^k = (-1)^k \frac{\partial}{\partial\lambda} \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)},$$

then replace the Gamma factors of large argument by their complete Stirling expansion.

We can now complete the proof of Theorem 2. It suffices simply to retain the terms corresponding to $j = 0$ in (65), in which case the error term becomes of the form $O((1-2q)^{-\gamma_0-1}\log(1-2q))$, which corresponds to a contribution that is $O(n^{\gamma_0}\log n) = O(n^{g-1}\log n)$ for PA_n.

Next, regarding the Fourier element of index $k = 0$, the function-to-coefficient correspondence yields

$$(1-2q)^{-\gamma_0-2} \quad \Longrightarrow \quad \frac{n^{\gamma_0+1}}{\Gamma(\gamma_0+2)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Thus, the coefficient κ_0 in (19) has value (cf. (62) and (63)) given by

$$\kappa_0 = \xi_0 \cdot p_0|_{q=1/2} \cdot \frac{1}{\Gamma(\gamma_0+2)}, \quad \gamma_0 = \frac{\log 3/2}{\log 2},$$

with ξ_0 as in (63). This, given the form (53) of p_k at $k = 0$, is equivalent to the value of κ_0 stated in Theorem 2 (where $g := \gamma_0 + 1 = \log_2 3$).

For a Fourier element of index $k \in \mathcal{Z}$, we have similarly

$$(1-2q)^{-\gamma_0-2-i\chi_k} \quad \Longrightarrow \quad \frac{n^{\gamma_0+1+i\chi_k}}{\Gamma(\gamma_0+2+i\chi_k)}\left(1+O\left(\frac{1}{n}\right)\right), \quad \text{where } \chi_k := \frac{2k\pi}{\log 2}.$$

We finally observe that

$$n^{\gamma_0+1+i\chi_k} = n^{\gamma_0+1}e^{i\chi_k \log n},$$

so that all the terms, for $k \in \mathbb{Z}$, are of the same asymptotic order (namely, $O(n^{\gamma_0+1})$) and their sum constitutes a Fourier series in $\log n$. The Fourier coefficient κ_k then satisfies, from the discussion above:

$$\kappa_k = \xi_0 \cdot p_k|_{q=1/2} \cdot \frac{1}{\Gamma(\gamma_0+2+i\chi_k)}.$$

Thus finally, with $g \equiv \gamma_0 + 1$:

$$\kappa_k = \frac{\pi}{9\log 2 \sin(\pi g + 2ik\pi^2/\log 2)\Gamma(g+1+2ik\pi/\log 2)} \prod_{j=0}^{\infty} \frac{(1-\frac{1}{3}2^{-j})(1-\frac{3}{2}2^{-j})}{(1-\frac{1}{2}2^{-j})^2}. \tag{72}$$

This completes the proof of Theorem 2. □

The same method shows the existence of a complete asymptotic expansion for PA_n.

Theorem 3. *The number of* 3*-sided prudent polygons satisfies a complete asymptotic expansion,*

$$PA_n \sim 2^n + 2^n \cdot n^g\left(\Xi_{0,0} + \frac{1}{n}(\log n \cdot \Xi_{1,1} + \Xi_{1,0}) + \frac{1}{n^2}\left(\log^2 n \cdot \Xi_{2,2} + \log n \cdot \Xi_{2,1} + \Xi_{2,0}\right)\cdots\right),$$

where $\Xi_{j,\ell}$ is an absolutely convergent Fourier series in $\log n$.

The non-oscillating form obtained by retaining only the constant terms of each Fourier series is computed by a symbolic manipulation system such as MAPLE or MATHEMATICA in a matter of seconds and is found to start as

$$\begin{aligned}\frac{\Omega_5}{2^n} \doteq{}& 1 + 0.1083842947 \cdot n^g + (-0.3928066917L + 0.5442458535) \cdot n^{g-1} \\ &+ \left(0.2627062704L^2 + 0.6950193894L + 0.6985601031\right) \cdot n^{g-2} \\ &+ \left(0.08310555463L^3 - 0.02188678892L^2 - 1.570478457L \right.\\ &\left.- 1.18810811075202\right) \cdot n^{g-3} \\ &+ \left(0.06722511293L^4 + 0.05494834609L^3 - 3.297513638L^2 - 4.663711650L \right.\\ &\left.- 4.156441653\right) \cdot n^{g-4},\end{aligned} \tag{73}$$

where $L = \log n$. In principle, all the coefficients have explicit forms in terms of the basic quantities that appear in Theorem 2 (augmented by derivatives of q-Pochhammer symbols at small rational values). However, the corresponding formulae blow up exponentially, so that we only mention here the next coefficient $-0.39280\ldots$ in (73), whose exact value turns out to be

$$-\kappa_0 \frac{\log 3}{\log^2 2} g.$$

Fig. 6 displays the difference between PA_n and the six-term extension Ω_6 of (73). It is piquant to note that all the terms given in Eq. (73) are needed in order to succeed in bringing the fluctuations

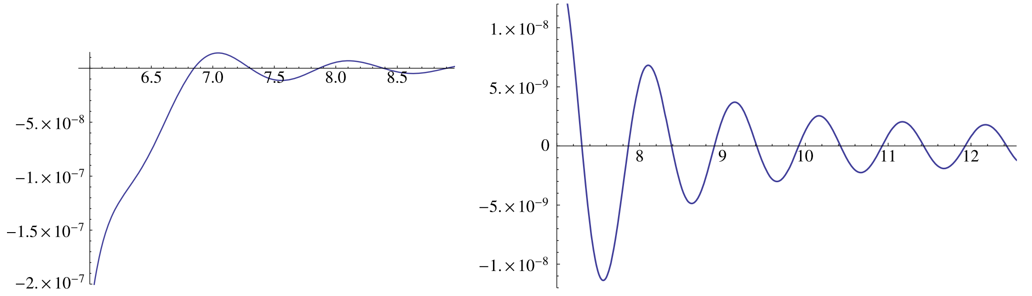

Fig. 6. *Left*: The difference $(PA_n - \Omega_6)2^{-n}n^{-g}$ against $\log_2 n$, where Ω_6 is the six-term extension of (73), for $n = 60, \dots, 500$; the plot reveals about three periods of the oscillating component in the asymptotic expansion of PA_n. *Right*: The corresponding plot, for values up to $n = 6000$. The diagrams confirm the presence of oscillations that, asymptotically, have amplitude of the order of 10^{-9}.

of Fig. 6 out of the closet. As a matter of fact, *before* the analysis of Theorem 3 was completed, the authors had tried empirically to infer the likely shape of the asymptotic expansion of PA_n, together with approximate values of coefficients, from a numerical analysis of series. The success was moderate and the conclusions rather "unstable" (leading on occasion to heated debates between coauthors). This state of affairs may well be present in other yet-to-be-analyzed models of combinatorics. It is pleasant that, thanks to complex asymptotic methods, eventually ... everything nicely fits in place.

4. Conclusions

Classification of prudent polygons. Our first conclusion is that the present study permits us to advance the classification of prudent walks and polygons: the generating functions and their coefficient asymptotics are now known in all cases up to 3-sided (walks by length; polygons by either perimeter or area). Functional equations are also known for 4-sided prudent walks and polygons, from which it is possible to distill plausible estimates. We can then summarize the present state of knowledge by the following table (compare with (4) and (5)).

Prudent polygons (by area)	Generating function	Asymptotic number	References
2-sided	rational	2^n	this paper, Section 2.2
3-sided	non-holonomic	$C_3(n) \cdot 2^n n^{\log_2 3}$	this paper, Theorem 2
4-sided	functional equation	$C_4(n) \cdot 2^n n^{1+\log_2 3}$ (?)	this paper, Section 2.4; Beaton et al. [1]

(74)

(The oscillating coefficient $C_3(n)$ is expressible in terms of the Fourier series $\kappa(u)$ of Theorem 2.) The numerical data relative to 4-sided polygons enumerated by area (last line) are from [1]: in all likelihood, the dominant singularity remains at $1/2$ and the critical exponent is $1 + \log_2 3$; that is, one more than the corresponding exponent for 3-prudent polygons. Examination of subdominant terms also suggests that the number of 4-polygons, once divided by n, satisfies an expansion of the form obtained in Theorem 3 for the 3-prudent counterparts. The "mean" amplitude is probably about 0.033, and, under the circumstances, there is little doubt that minuscule oscillations (rendered by $C_4(n)$) must also be present. The foregoing analysis of the 3-prudent case then at least has the merit of pointing towards the type of singularity to be expected for 4-prudent polygons as well as, possibly, to methods of attack for this case.

Methodology. The generating function of 3-prudent polygons has been found to be a q-hypergeometric function, with the argument and parameters subject to a rational substitution. The methods developed here should clearly be useful in a number of similar situations. Note that the asymptotic enumeration of prudent walks and polygons by length and perimeter is in a way easier since the *dominant* singularity is polar or algebraic. (Bousquet-Mélou [3] however exhibits an interesting situation where the *complete* singular structure has a complex geometry.)

Estimates involving periodic oscillations are not unheard of in combinatorial asymptotics [20,28, 29]. What is especially interesting in the case of 3-sided prudent polygons is the pattern of singularities that accumulate geometrically fast to the dominant singularity. This situation is prototypically encountered in the already evoked problem of the longest run in strings: the classical treatment is via real analytic methods followed by a Mellin analysis of the expressions obtained; see [25]. In fact, the chain

$$\text{Coefficient asymptotics} \leadsto \text{Mellin transform} + \text{Singularity analysis} \tag{75}$$

is applicable for moment analyses. For instance, the analysis of the expected longest run of the letter a in a random binary sequence over the alphabet $\{a, b\}$ leads to the generating function [20, Example V.4]

$$\Phi(z) = (1-z)\sum_{k\geqslant 0}\frac{z^k}{1-2z+z^{k+1}},$$

to which the chain (75) can be applied.

Another source of similar phenomena is the analysis of digital trees [26,35], when these are approached via ordinary generating functions (rather than the customary exponential or Poisson generating functions). Typically, in the simplest case of node-depth in a random digital tree, one encounters the generating function

$$\Psi(z) = \frac{1}{1-z}\sum_{k\geqslant 0}\frac{2^{-k}}{1-z(1-2^{-k})},$$

where the geometric accumulation of poles towards 1 is transparent, so that the chain (75) can once more be applied [19].

We should finally mention that "critical" exponents similar to $g = \log_2 3$ surface at several places in mathematics, usually accompanied by oscillation phenomena, but they do so for reasons essentially simpler than in our chain (75). For instance, in fractal geometry, the Hausdorff dimension of the triadic Cantor set is $1/g$, see [14], while that of the familiar Sierpiński gasket is g, so that g occurs as critical exponent in various related integer sequences [18]. The same exponent $g = \log_2 3 \approx 1.58$ is otherwise known to most students of computer science, since it appears associated to the recurrent sequence $f_n = n + 3f_{\lfloor n/2\rfloor}$, which serves to describe the complexity of Karatsuba multiplication [27] (where, recursively, the product of *two* double-precision numbers is reduced to *three* single-precision numbers). In such cases, the exponent g is eventually to be traced to the singularity (at $s = g$, precisely) of the Dirichlet series

$$\omega(s) = \frac{1}{1-3\cdot 2^{-s}},$$

which is itself closely related to the Mellin transforms of our Eq. (50). See also the studies by Drmota and Szpankowski [10], Dumas [12], as well as [18] for elements of a general theory.

For the various reasons evoked above, we believe that the asymptotic methods developed in the present paper are of a generality that goes somewhat beyond the mere case of 3-sided prudent polygons, and may illuminate other models of interest in lattice combinatorics and statistical mechanics.

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