THE CRITICAL FUGACITY FOR SURFACE ADSORPTION OF SELF-AVOIDING WALKS ON THE HONEYCOMB LATTICE IS $1+\sqrt{2}$

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Abstract. In 2010, Duminil-Copin and Smirnov proved a long-standing conjecture of Nienhuis, made in 1982, that the growth constant of self-avoiding walks on the hexagonal (a.k.a. honeycomb) lattice is $\mu = \sqrt{2+\sqrt{2}}$. A key identity used in that proof was later generalised by Smirnov so as to apply to a general $O(n)$ loop model with $n \in [-2, 2]$ (the case $n = 0$ corresponding to self-avoiding walks).

We modify this model by restricting to a half-plane and introducing a surface fugacity y associated with boundary sites (also called surface sites), and obtain a generalisation of Smirnov's identity. The critical value of the surface fugacity was conjectured by Batchelor and Yung in 1995 to be $y_c = 1 + 2/\sqrt{2-n}$. This value plays a crucial role in our generalized identity, just as the value of growth constant did in Smirnov's identity.

For the case $n = 0$, corresponding to self-avoiding walks interacting with a surface, we prove the conjectured value of the critical surface fugacity. A crucial part of the proof involves demonstrating that the generating function of self-avoiding bridges of height T, taken at its critical point $1/\mu$, tends to 0 as T increases, as predicted from SLE theory.

1. Introduction

The n-vector model, also called $O(n)$ model, introduced by Stanley in 1968 [25] is described by the Hamiltonian

$$\mathcal{H}(d,n) = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j,$$

where d denotes the dimensionality of the lattice, i and j are adjacent sites, and $\mathbf{s}_i$ is an n-dimensional vector of magnitude $\sqrt{n}$. Writing $x = J/k_B T$, the corresponding partition function of this model on a two-dimensional square domain with N^2 sites is given by

$$Z_{N^2}(x) = \int \prod_k \mathrm{d}\mu(\mathbf{s}_k) \prod_{\langle i,j \rangle} w_{ij}, \qquad w_{ij} = \mathrm{e}^{x \mathbf{s}_i \cdot \mathbf{s}_j}, \tag{1}$$

where μ is the spherical measure on the $(n-1)$-dimensional sphere of radius $\sqrt{n}$, normalised by $\int \mathrm{d}\mu(\mathbf{s}) = 1$.

When $n = 1$ the Hamiltonian above describes the Ising model, and when $n = 2$ it describes the classical XY model. Two other interesting limits, which leave a lot to be desired from a pure mathematical perspective, are the limit $n \to 0$, in which case one recovers the self-avoiding walk (SAW) model, as first pointed out by de Gennes [9]; and the limit $n \to -2$, corresponding to random walks, or more generally to a free-field Gaussian model, as shown by Balian and Toulouse [2].

Self-avoiding walks will be central in this paper. They have been considered as models of long-chain polymers in solution since the middle of the last century — see for example articles by Orr [22] and Flory [14]. Since that time they have been studied and extended by polymer chemists as models of polymers; by mathematicians as combinatorial models of pristine simplicity in their description, yet malevolent difficulty in their solution; by computer scientists interested in computational complexity; and by biologists using them to model properties of DNA and other biological polymers of interest.

Date: September 25, 2013.

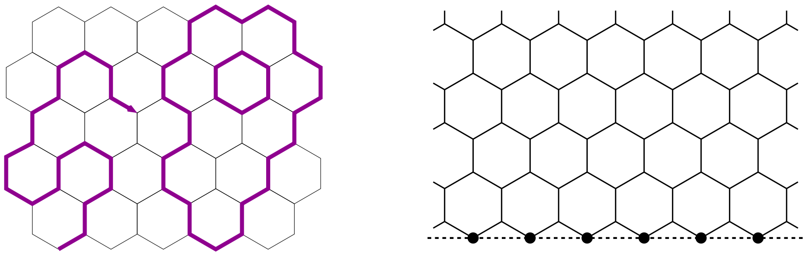

FIGURE 1. *Left*: A configuration of the loop model on the honeycomb lattice. *Right*: The half-plane and its boundary.

It is argued in [10] that the critical behaviour of the n-vector model is unchanged if the Boltzmann weight w_{ij} in (1) is replaced by $w_{ij} = 1 + x\,\mathbf{s}_i \cdot \mathbf{s}_j$. Moreover, this new model is equivalent to a loop model with a weight n attached to closed loops [10]:

$$Z_{N^2}(x) = \sum_{\gamma} x^{|\gamma|} n^{\ell(\gamma)},$$

where (on the honeycomb lattice) γ is a configuration of non-intersecting loops, $|\gamma|$ is the number of edges and $\ell(\gamma)$ is the number of loops. We call this model the $O(n)$ *loop model*. In the following we consider a loop model with a defect, *i.e.*, a model of closed loops with one self-avoiding walk component[1]. A typical configuration is shown in Fig. 1, left.

In 1982 Nienhuis [21] showed that, for $n \in [-2, 2]$, the loop model on the honeycomb lattice could be mapped onto a solid-on-solid model, from which he was able to derive the critical points and critical exponents, subject to some plausible assumptions. These results agreed with the known exponents and critical point for the Ising model, and they predicted exact values for those models corresponding to other values of the spin dimensionality n. In particular, for $n = 0$ the critical point for the honeycomb lattice SAW model was predicted to be $x_c = 1/\sqrt{2+\sqrt{2}}$.

This result was finally proved 28 years later by Duminil-Copin and Smirnov [12]. The starting point of their proof is a *local* identity for a "parafermionic" observable, valid at every vertex of the lattice. Then they obtain a *global* identity linking several walk generating functions by summing over all vertices of a domain[2]. Smirnov [24] then extended the local identity to the general honeycomb $O(n)$ loop model with $n \in [-2, 2]$. This extension provides an alternative way of predicting the value of the critical point $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$ as conjectured by Nienhuis.

Nienhuis's results were concerned with bulk systems. Interesting surface phenomena can also be studied if one considers the $O(n)$ loop model in a half-space, with vertices in the surface (the boundary of the half-space) having an associated fugacity. See Fig. 1, right. The partition function becomes

$$Z_{N^2}(x, y) = \sum_{\gamma} x^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)}, \tag{2}$$

where $c(\gamma)$ is the number of vertices on the boundary occupied by γ. The value of the fugacity y can be changed to result in a repulsive or attractive interaction with the surface, with a phase transition occurring at the point distinguishing between these two regimes. In the limit of a large lattice, the free energy per site may be decomposed as

$$f_{N^2}(x, y) := -\frac{k_B T}{N^2} \log Z_{N^2}(x, y) = f_{\text{bulk}}(x, y) + \frac{1}{N} f_{\text{surface}}(x, y) + \cdots,$$

and surface phase transitions correspond to singularities in f_{surface}. At $x = x_c$, the adsorption transition is an example of a *special* surface transition [8].

[1] Defects correspond to correlation functions of the underlying spin model. It follows that the critical point remains the same.

[2] A more formal presentation of their proof has been provided by Klazar [18].

In 1995 Batchelor and Yung [4] extended Nienhuis's work to the adsorption problem described above. Using the integrability of an underlying lattice model and comparison to numerical results, they conjectured the value of the critical surface fugacity for the honeycomb lattice $O(n)$ loop model.[3]

Conjecture 1 (**Batchelor and Yung**). *For the $O(n)$ loop model on the semi-infinite hexagonal lattice of Fig. 1 with $n \in [-2, 2]$, associate a fugacity $x_c(n) = 1/\sqrt{2+\sqrt{2-n}}$ with occupied vertices and an additional fugacity y with occupied vertices on the boundary. Then the model undergoes a surface transition at*

$$y = y_c(n) = 1 + \frac{2}{\sqrt{2-n}}.$$

In this paper we first show that the local identity proved by Smirnov [24] for the $O(n)$ loop model can be generalised to a half-plane system with a surface fugacity (Lemma 3). We use this to prove a generalisation of the global identity of Duminil-Copin and Smirnov including a surface fugacity (Proposition 4). The contribution of one of these generating functions vanishes at $y = y_c(n)$, which lends support to the above conjecture.

We then focus on the case $n = 0$, corresponding to self-avoiding walks interacting with an impenetrable surface. This case is somewhat degenerate since no loops are allowed and we therefore adopt the definition of $y_c(0)$ given by Hammersley, Torrie and Whittington [15], which we recall in Section 3.1. With this definition, we prove Conjecture 1 for $n = 0$: a self-avoiding walk is adsorbed if $y > 1 + \sqrt{2}$ and desorbed if $y < 1 + \sqrt{2}$.

Theorem 2. *The critical surface fugacity for self-avoiding walks on the honeycomb lattice is*

$$y_c(0) = 1 + \sqrt{2}.$$

The proof of Theorem 2 relies of course on our global identity, but also requires earlier results dealing with SAWs confined to a half-plane or a strip: notably, existence of the critical value of the surface fugacity y, enumeration of SAWs in a strip and the behaviour as the size of the strip increases, among others. Most of these results have been proved for the square (and hypercubic) lattice, but we need to adapt these proofs to the honeycomb case, which we do in Section 3. Section 4 combines these results and the global identity to prove Theorem 2. A third key ingredient, of independent interest, is that the generating function of *bridges* of height T, taken at x_c, tends to 0 as T increases. The proof is probabilistic in nature, and is given in the appendix.

To conclude this introduction, let us mention that one can also consider a honeycomb half-plane with a *vertical* (rather than horizontal) boundary. The techniques of this paper have been adapted by the first author to determine the critical surface fugacity in this case [5], which was conjectured by Batchelor, Bennett-Wood and Owczarek [3]. For other lattices, we do not have conjectures for the values of the critical fugacities; instead, numerical estimates using series analysis and Monte Carlo methods are the best current results. New methods of estimating the growth constants and critical surface fugacities of the square and triangular lattices, inspired by results presented in [12] and this paper, are explored in [6, 7].

2. Smirnov's identity in the presence of a boundary

We consider the honeycomb lattice, embedded in the complex plane $\mathbb{C}$ in such a way that the edges have unit length. This allows us to consider vertices of the lattice as complex numbers. It is also convenient to start and end self-avoiding walks at a mid-edge of the lattice. (That is, the point on an edge precisely halfway between its two incident vertices.) We restrict the lattice to a half-plane, bounded from above by a horizontal surface consisting of *weighted sites* (Fig. 2).

[3]Batchelor and Yung use different notation, and in particular weight their configurations slightly differently. They use weights t_b and t_s (corresponding to bulk and surface edges respectively), and the correspondence with our notation is $x = 1/t_b$ and $y = t_b/t_s$.

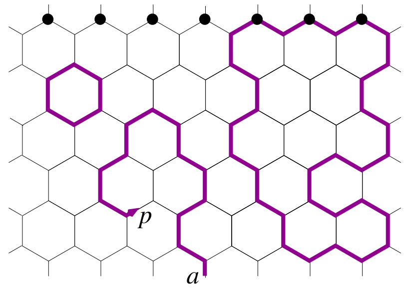

FIGURE 2. A configuration γ on a finite domain, with the weighted vertices on the top boundary indicated. The contribution of γ to $F(p)$ is $\mathrm{e}^{-5i\sigma\pi/3}x^{51}y^3n^2$.

We further consider a domain D of this half-lattice, consisting of a finite connected collection of half-edges such that for every vertex v incident to at least one half-edge of D, all three half-edges incident to v actually belong to D. We denote by $V(D)$ the set of vertices incident to half-edges of D. Those mid-edges of D which are adjacent to only one vertex in $V(D)$ form the *boundary* ∂D. A *configuration* γ consists of a (single) self-avoiding walk w and a (finite) collection of closed loops, which are self-avoiding and do not meet one another nor w. We denote by $|\gamma|$ the number of vertices occupied by γ (also called the *length*), by $c(\gamma)$ the number of *contacts* with the surface (*i.e.* vertices of the surface occupied by γ), and by $\ell(\gamma)$ the number of loops. See Fig. 2 for an example.

Let a be a fixed mid-edge on the boundary ∂D. For any mid-edge p of D, define the following *generating function*, or *observable*:

$$F(p;x,y,n,\sigma) \equiv F(p) := \sum_{\gamma:a\rightsquigarrow p} x^{|\gamma|}y^{c(\gamma)}n^{\ell(\gamma)}\mathrm{e}^{-i\sigma W(w)}, \tag{3}$$

where the sum is over all configurations γ in D for which the SAW component w runs from a to p and $W(w)$ is the *winding angle* of w, that is, $\pi/3$ times the difference between the number of left turns and the number of right turns.

The case $y = 1$ of the following lemma is due to Smirnov [24].

Lemma 3 (**The local identity**). *For $n \in [-2, 2]$, set $n = 2\cos\theta$ with $\theta \in [0, \pi]$. Let*

$$\sigma = \frac{\pi - 3\theta}{4\pi}, \qquad x^{-1} = x_{\mathrm{c}}^{-1} := 2\cos\left(\frac{\pi+\theta}{4}\right) = \sqrt{2-\sqrt{2-n}}, \qquad \textit{or} \tag{4}$$

$$\sigma = \frac{\pi + 3\theta}{4\pi}, \qquad x^{-1} = x_{\mathrm{c}}^{-1} := 2\cos\left(\frac{\pi-\theta}{4}\right) = \sqrt{2+\sqrt{2-n}}. \tag{5}$$

Then for a vertex $v \in V(D)$ not belonging to the weighted surface, the observable F defined by (3) *satisfies*

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0, \tag{6}$$

where p, q, r are the mid-edges adjacent to v.

If $v \in V(D)$ lies on the weighted surface,

$$\begin{aligned}(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = \\ (q-v)(1-y)(x_{\mathrm{c}}y\lambda)^{-1} \sum_{\gamma:a\rightsquigarrow q,p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}n^{\ell(\gamma)}\mathrm{e}^{-i\sigma W(w)} \\ + (r-v)(1-y)(x_{\mathrm{c}}y\bar{\lambda})^{-1} \sum_{\gamma:a\rightsquigarrow r,p} x_{\mathrm{c}}^{|\gamma|}y^{c(\gamma)}n^{\ell(\gamma)}\mathrm{e}^{-i\sigma W(w)},\end{aligned} \tag{7}$$

where $\lambda = \mathrm{e}^{-i\sigma\pi/3}$ is the weight accrued by a walk for each left turn, p, q, r are the three mid-edges adjacent to v, taken in counterclockwise order, with p just above v, and the first (resp. second) sum runs over configurations γ whose SAW component w goes from a to p via q (resp. via r).

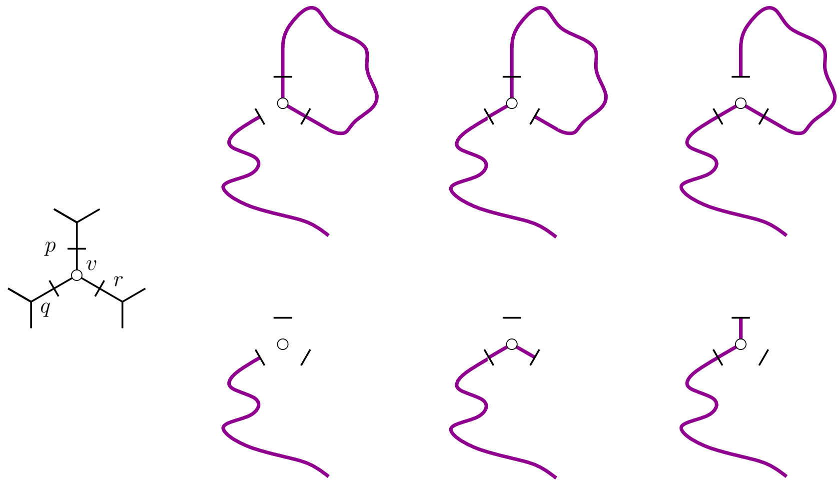

FIGURE 3. Two groups of configurations ending at a mid-edge adjacent to the vertex v. The contribution of each group to (6) is 0.

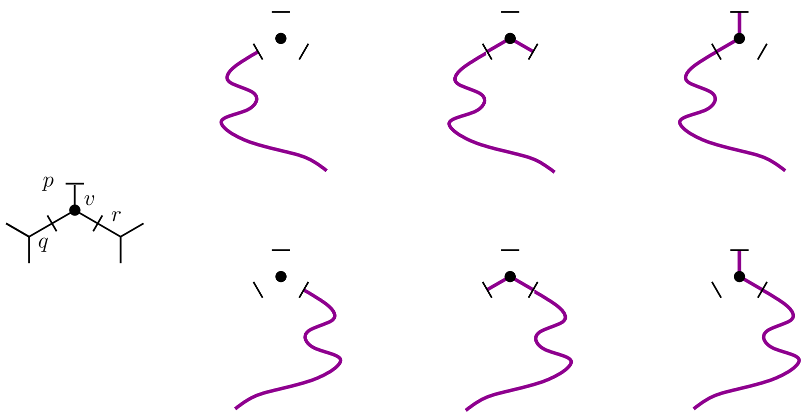

FIGURE 4. Two groups of walks ending at a mid-edge adjacent to a surface vertex. The top group leads to (8), and the bottom group to (9).

Equation (4) corresponds to the larger of the two special values of the step weight x and corresponds to a point in the dense regime where the model turns out to be integrable. Equation (5) gives the value of the critical point, separating the dense and dilute phases. Both special values of x were predicted by Nienhuis [21] from a renormalisation group analysis, the first value corresponding to a stable fixed point and the latter to an unstable one. In what follows, when we refer to the dense and dilute regimes, we mean the regimes with values of the step weight x given by (4) and (5) respectively.

Proof. If v does not belong to the surface, the proof is completely analogous to the proof of Lemma 4 in [24]: One observes that the left-hand side of (6) counts (weighted) configurations ending at a mid-edge adjacent to v, and organizes these configurations by groups of three, as shown in Fig. 3 (which, up to rotations, includes all possible cases). It is then easy to check that, for the given values of σ and x_c, the contribution of each group vanishes. The fact that $y \neq 1$ in our paper makes no difference, because the number of weighted vertices is the same for all walks in a group.

This is not true if v belongs to the surface. Still, let us determine the contribution of each group. We first note that groups of the first type (for which the three mid-edges p, q, and r are visited) cannot exist when v is on the surface. For groups of the second type, we distinguish two cases, depending on whether the walk approaches v via q or via r (Fig. 4). If the leftmost configuration in each group of Fig. 4 is denoted γ_1, and the rightmost one γ, with associated SAW components w_1 and w, then the contribution in the first case is

$$(q-v)x_{\mathrm{c}}^{|\gamma_1|}y^{c(\gamma_1)}n^{\ell(\gamma_1)}e^{-i\sigma W(w_1)}(1+x_{\mathrm{c}}y\bar{\lambda}j+x_{\mathrm{c}}y\lambda\bar{j}) \tag{8}$$

with $j=e^{2i\pi/3}$. But we know that this vanishes when $y=1$ (this is Smirnov's result), so the last term in parentheses must be $(1-y)$. Moreover,

$$|\gamma_1|=|\gamma|-1, \quad c(\gamma_1)=c(\gamma)-1, \quad \ell(\gamma_1)=\ell(\gamma), \quad W(w_1)=W(w)-\pi/3,$$

and one concludes that groups of walks approaching v via q give the first sum in (7). Similarly, for a group of walks approaching v via r, the contribution is

$$(r-v)x_{\mathrm{c}}^{|\gamma_1|}y^{c(\gamma_1)}n^{\ell(\gamma_1)}e^{-i\sigma W(w_1)}(1+x_{\mathrm{c}}y\bar{j}\lambda+x_{\mathrm{c}}yj\bar{\lambda})= \\ (r-v)(1-y)x_{\mathrm{c}}^{|\gamma|-1}y^{c(\gamma)-1}n^{\ell(\gamma)}e^{-i\sigma(W(w)+\pi/3)}, \tag{9}$$

which gives the second sum in (7). ■

In [12], Duminil-Copin and Smirnov prove and use Lemma 3 to prove that the growth constant of the self-avoiding walk is given by the case $n=0$ of the dilute regime (5): $x_{\mathrm{c}}^{-1}:=2\cos(\pi/8)=\sqrt{2+\sqrt{2}}$. They do so by considering a special trapezoidal domain $D_{L,T}$ as shown in Fig. 5, and deriving from the local identity a global identity that relates several generating functions counting walks in this domain. Here we generalise this identity to a general $O(n)$ model including a boundary weight.

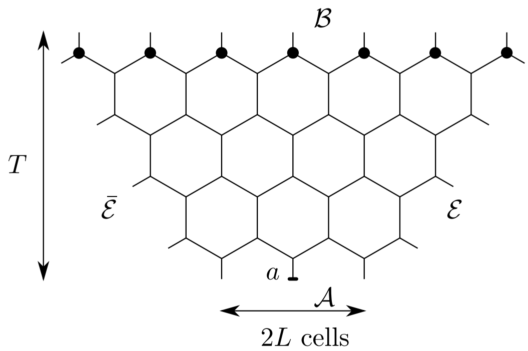

FIGURE 5. Finite patch $D_{T,L}$ of the half hexagonal lattice, with $T=4$ and $L=1$ (the convention on T is chosen in such a way a walk of minimal length going from the bottom to the top of the domain contains $T-1$ vertical edges and two vertical half-edges, one at each end of the walk). The SAW components of configurations start on the central mid-edge a of the bottom boundary. The weighted vertices, belonging to the surface, are marked with a black disc.

We partition the boundary $\partial D_{T,L}$ into four subsets $\mathcal{A}$, $\mathcal{B}$, $\bar{\mathcal{E}}$ and $\mathcal{E}$ as illustrated in Fig. 5. We also define four generating functions, counting configurations in $D_{T,L}$ starting from a and ending in $\partial D_{T,L}$. First,

$$A_{T,L}(x,y):=\sum_{\gamma:a\rightsquigarrow\mathcal{A}\setminus\{a\}} x^{|\gamma|}y^{c(\gamma)}n^{\ell(\gamma)}, \tag{10}$$

where the sum is over all configurations in $D_{T,L}$ whose SAW component goes from the mid-edge a to a mid-edge of $\mathcal{A}\setminus\{a\}$. We similarly define the generating functions $A^{\circ}_{T,L}(x,y)$, $B_{T,L}(x,y)$ and $E_{T,L}(x,y)$ for configurations ending in $\{a\}$, $\mathcal{B}$, and $\bar{\mathcal{E}}\cup\mathcal{E}$ respectively. Note that configurations counted by A° comprise *only* closed loops inside $D_{T,L}$; that is, their SAW component is the empty walk $a\rightsquigarrow a$.

Proposition 4. *For $n = 2\cos\theta$ and $x_{\rm c}^{-1} := 2\cos((\pi \pm \theta)/4)$, the above defined generating functions satisfy*

$$A^{\circ}_{T,L}(x_{\rm c}, y) = \cos\left(\frac{3(\pi \pm \theta)}{4}\right) A_{T,L}(x_{\rm c}, y) + \cos\left(\frac{\pi \pm \theta}{2}\right) E_{T,L}(x_{\rm c}, y) + \frac{y^* - y}{y(y^* - 1)} B_{T,L}(x_{\rm c}, y), \tag{11}$$

where

$$y^* = \frac{1}{1 - 2x_{\rm c}^2} = 1 \mp \frac{2}{\sqrt{2 - n}}.$$

Observe that in the dilute case $x_{\rm c}^{-1} = 2\cos((\pi - \theta)/4)$, the value of y^* coincides with the predicted value of $y_{\rm c}(n)$ given in Conjecture 1. In Section 4, we use the above identity to prove Conjecture 1 in the case $n = 0$ (that is, Theorem 2). In this case the left-hand side of (11) reduces to 1, all coefficients are positive as long as $y < y^*$, so that the polynomials $A_{T,L}$, $B_{T,L}$ and $E_{T,L}$ are uniformly bounded, independently of T and L. Just as in the proof of Duminil-Copin and Smirnov for the growth constant of SAWs, the bound on $B_{T,L}$ is an important ingredient of our proof. The identity (11) allows $B_{T,L}(x_{\rm c}, y)$ to diverge for $y \geq y^*$ (as T and L grow) which signals the surface transition at the $\mathcal{B}$ boundary.

Proof. Let p_v, q_v, r_v be the mid-edges adjacent to a vertex v. Let F be the observable defined by (3), and take $\sigma = (\pi \mp 3\theta)/(4\pi)$ as in Lemma 3. We compute the sum

$$S := \sum_{v \in V(D_{T,L})} \big((p_v - v)F(p_v) + (q_v - v)F(q_v) + (r_v - v)F(r_v)\big) \tag{12}$$

in two ways.

Firstly, all summands of (12) associated with a non-weighted vertex v are 0 by the first part of Lemma 3. We are left with the contribution of vertices lying on the surface, given in the second part of the lemma. Since $W(w) = 0$ for all walks occurring in (7),

$$2S = e^{-5i\pi/6}(1 - y)(x_{\rm c} y \lambda)^{-1} \sum_{p \in \mathcal{B}, \gamma : a \rightsquigarrow q, p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} + e^{-i\pi/6}(1 - y)(x_{\rm c} y \bar{\lambda})^{-1} \sum_{p \in \mathcal{B}, \gamma : a \rightsquigarrow r, p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)},$$

where q (resp. r) stands for the SW (resp. SE) mid-edge adjacent to v. The factor 2 accounts for the fact that edges have length 1, so that terms like $(p - v)$ have modulus 1/2. Now reflecting a configuration γ that reaches a mid-edge $p \in \mathcal{B}$ from the SW gives a configuration γ' that reaches a mid-edge $p' \in \mathcal{B}$ from the SE. Moreover, $|\gamma| = |\gamma'|$, $c(\gamma) = c(\gamma')$ and $\ell(\gamma) = \ell(\gamma')$. Hence

$$\begin{aligned} 2S &= (1 - y)(x_{\rm c} y)^{-1} \sum_{p \in \mathcal{B}, \gamma : a \rightsquigarrow q, p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(e^{-5i\pi/6}\bar{\lambda} + e^{-i\pi/6}\lambda\right) \\ &= -2i(1 - y)(x_{\rm c} y)^{-1} \cos\left(\frac{\pi \pm \theta}{4}\right) \sum_{p \in \mathcal{B}, \gamma : a \rightsquigarrow q, p} x_{\rm c}^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \\ &= -i(1 - y)(x_{\rm c} y)^{-1} \cos\left(\frac{\pi \pm \theta}{4}\right) B_{T,L}(x_{\rm c}, y) \qquad \text{by symmetry} \\ &= -\frac{i}{2}(1 - y)(x_{\rm c}^2 y)^{-1} B_{T,L}(x_{\rm c}, y). \end{aligned} \tag{13}$$

To obtain another expression for S, starting from (12), note that any mid-edge p not belonging to $\partial D_{T,L}$ contributes to two terms in the sum, for vertices v_1 and v_2, and these two terms cancel because $(p - v_1) = -(p - v_2)$. Thus we are left with precisely the contributions of those mid-edges in $\partial D_{T,L}$:

$$2S = -i \sum_{p \in \mathcal{A}} F(p) + e^{-5i\pi/6} \sum_{p \in \bar{\mathcal{E}}} F(p) + e^{-i\pi/6} \sum_{p \in \mathcal{E}} F(p) + i \sum_{p \in \mathcal{B}} F(p). \tag{14}$$

We again use symmetry arguments to rewrite this sum. First, denoting $\mathcal{A} = \{a\} \cup \mathcal{A}^- \cup \mathcal{A}^+$ (with $\mathcal{A}^-$ to the left of a), we have

$$\begin{aligned}\sum_{p\in\mathcal{A}} F(p) &= A^{\circ}_{T,L}(x_c, y) + \sum_{\gamma: a\rightsquigarrow\mathcal{A}^-} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(\lambda^3 + \bar\lambda^3\right) \\ &= A^{\circ}_{T,L}(x_c, y) - \cos\left(\frac{3(\pi \pm \theta)}{4}\right) A_{T,L}(x_c, y).\end{aligned}$$

Similarly,

$$\begin{aligned}e^{-i\pi/3}\sum_{p\in\bar{\mathcal{E}}} F(p) + e^{i\pi/3}\sum_{p\in\mathcal{E}} F(p) &= \sum_{\gamma: a\rightsquigarrow\bar{\mathcal{E}}} x_c^{|\gamma|} y^{c(\gamma)} n^{\ell(\gamma)} \left(e^{-i\pi/3}\lambda^2 + e^{i\pi/3}\bar\lambda^2\right) \\ &= -\cos\left(\frac{\pi \pm \theta}{2}\right) E_{T,L}(x_c, y).\end{aligned}$$

Finally,

$$\sum_{p\in\mathcal{B}} F(p) = B_{T,L}(x_c, y).$$

Equating (13) and (14) gives the proposition. ■

3. Confined self-avoiding walks

In the remainder of this paper we specialise to $n = 0$, corresponding to self-avoiding walks. In this case, we will prove that the critical surface fugacity is $y_c = 1 + \sqrt{2}$. In this section we first review some basic but important background, and then adapt to the honeycomb lattice some known results about *square lattice* SAWs confined to a half-plane or a strip. These results will be used in Section 4, where we prove our main result.

Again, we consider SAWs on the honeycomb lattice, starting and ending at a mid-edge. The simplest model associates a fugacity x with each visited vertex (or *step*, or *monomer*). One then studies the generating function

$$C(x) = \sum_{k\geq 0} c_k x^k,$$

where c_k is the number of SAWs of k monomers, considered equivalent up to a translation. A simple concatenation argument and a classical lemma on sub-multiplicative sequences suffice to prove that the *growth constant*

$$\mu := \lim_{k\to\infty} (c_k)^{1/k}$$

exists and is finite [20, Chap. 1]. Of course, $1/\mu$ is the radius of convergence x_c of the series $C(x)$. Duminil-Copin and Smirnov [12] proved Nienhuis's conjecture [21] that, for the honeycomb lattice, $\mu = \sqrt{2 + \sqrt{2}}$.

3.1. Self-avoiding walks in a half-plane

We now consider SAWs in the upper half-plane, originating at a mid-edge a just below the surface (Fig. 6). It is known that the growth constant for such walks is the same as for the bulk case (see [27] or [20, Chap. 3]). We also add a fugacity y to vertices in the surface. In physics terms, $y = \mathrm{e}^{-\epsilon/k_B T}$ where ϵ is the energy associated with a surface vertex, T is the absolute temperature and k_B is Boltzmann's constant.

We now consider the following partition function (which is a polynomial in y):

$$c_k^+(y) := \sum_{|w|=k} y^{c(w)},$$

where the sum runs over half-plane SAWs w of length k and $c(w)$ denotes the number of *contacts* of w with the surface (*i.e.*, the number of vertices of the surface visited by w).

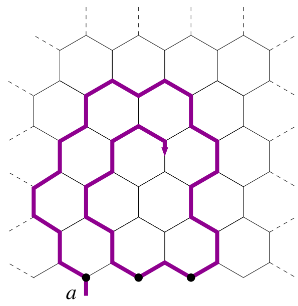

FIGURE 6. A self-avoiding walk in a half-plane, with weights attached to the vertices of the surface (indicated by black discs).

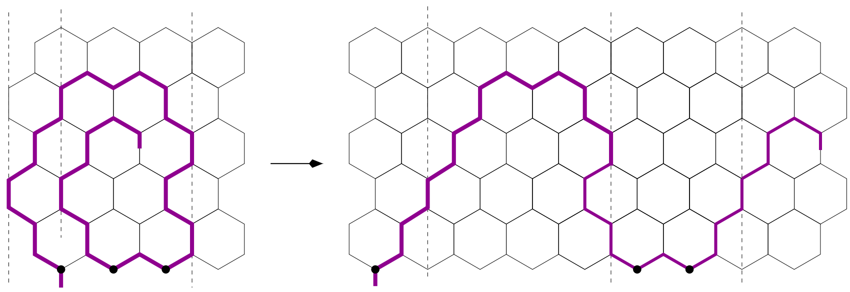

FIGURE 7. Unfolding a half-plane SAW on the honeycomb lattice.

Proposition 5. *For $y > 0$,*

$$\mu(y) := \lim_{k\to\infty} c_k^+(y)^{1/k}$$

exists and is finite. It is a log-convex, non-decreasing function of $\log y$*, and therefore continuous and almost everywhere differentiable.*

For $0 < y \leq 1$,

$$\mu(y) = \mu(1) = \mu.$$

Moreover, for any $y > 0$,

$$\mu(y) \geq \max(\mu, \sqrt{y}).$$

This behaviour implies the existence of a critical value y_c, with $1 \leq y_c \leq \mu^2$, such that

$$\mu(y) \begin{cases} = \mu & \text{if } y \leq y_c, \\ > \mu & \text{if } y > y_c. \end{cases}$$

Proof. The existence of $\mu(y)$ has been proved by Hammersley, Torrie and Whittington [15] in the case of the d-dimensional hypercubic lattice. Their discussion and proof, which use concatenation and *unfolding* of walks, apply *mutatis mutandis* to the honeycomb lattice. Unfolding consists of reflecting parts of the walk in vertical lines passing through those vertices of the walk with maximal and minimal x-coordinates (Fig. 7). This unfolding is repeated until the origin and end-point have minimal and maximal x-coordinates respectively. The main advantage of such unfolded walks is that they can be concatenated without creating self-intersections (this may require the addition of a few steps between the walks).

The other results are elementary, and adapted from an earlier paper of Whittington [27]. In particular, the lower bound $\mu(y) \geq \sqrt{y}$ is obtained by counting zig-zag walks sticking to the surface. ■

The function $\mu(y)$ is not known explicitly, but the main result of this paper is that $y_c = 1 + \sqrt{2} = \mu^2 - 1$. The behaviour of $\mu(y)$ as $y \to \infty$ has recently been established by Rychlewski and Whittington [23], who proved that, on the square lattice, $\mu(y)$ is asymptotic to y. This translates into $\mu(y) \sim \sqrt{y}$ in our honeycomb setting.

The critical value y_c, which we have defined in analytic terms, can also be given a probabilistic description. Fix k, and assign to each half-plane SAW w of length k the probability

$$\frac{y^{c(w)}}{c_k^+(y)}.$$

If y is large, this probability distribution favours walks with many contacts, while if y is small, the walk is repelled by the surface. The mean density of vertices of the walk lying in the surface is

$$\frac{1}{k\,c_k^+(y)} \sum_{|w|=k} c(w) y^{c(w)} = \frac{y}{k} \frac{\partial \log c_k^+(y)}{\partial y}.$$

Recall that $\frac{1}{k} \log c_k^+(y)$ tends to $\log \mu(y)$ as $k \to \infty$. In the limit of infinitely long walks, it can be shown[4] that the above density tends to

$$y \frac{\partial \log \mu(y)}{\partial y}.$$

From the behaviour of $\mu(y)$ given in Proposition 5, we see that the density of vertices on the surface is 0 for $y < y_c$ and is positive for $y > y_c$. In other words, the critical value y_c distinguishes between the *desorbed* and *adsorbed* phases.

3.2. Self-avoiding walks in a strip

As discussed in the previous subsection, the usual model of surface-interacting polymers considers walks originating in a surface and interacting with monomers (or edges) in that surface. One way to study such systems is to consider interacting walks in a strip, and then to take the limit as the strip width becomes infinite. Clearly, if one studies walks in a strip, it is possible to consider interactions with both the top and bottom surface. The results of this section will reconcile the (apparently inconsistent) settings of Section 2 (where weighted vertices are at the top of the domain, opposite the starting point) and Section 3.1 (where weighted vertices lie at the bottom of the domain, on the same side as the starting point).

Consider a strip of height T on the honeycomb lattice, as shown in Fig. 8. We consider SAWs that originate at a mid-edge a just below the bottom of the strip. Such walks are said to be *arches* if they end at the bottom of the strip, and *bridges* if they end at the top (Fig. 8). We now consider the bivariate polynomials

$$c_{T,k}(y,z) = \sum_{|w|=k} y^{bc(w)} z^{tc(w)},$$

where the sum runs over all SAWs w of length k in the T-strip and $bc(w)$ and $tc(w)$ are the numbers of contacts of w with the bottom and top of the strip respectively. We define similar polynomials $a_{T,k}(y,z)$and $b_{T,k}(y,z)$ for arches and bridges.

Proposition 6. *For $y, z > 0$, one has*

$$\lim_{k\to\infty} a_{T,k}(y,z)^{1/k} = \lim_{k\to\infty} b_{T,k}(y,z)^{1/k} = \lim_{k\to\infty} c_{T,k}(y,z)^{1/k} := \mu_T(y,z),$$

where $\mu_T(y,z)$ is finite, and non-decreasing in y and z. By the symmetry of bridges,

$$\mu_T(y,z) = \mu_T(z,y),$$

and so, in particular, $\mu_T(y,1) = \mu_T(1,y)$. Finally, $\mu_T(1,y)$ is a log-convex and thus continuous function of $\log(y)$.

[4] The exchange of the limit and the derivative is possible thanks to the convexity of $\log \mu(y)$, see for instance [26, Thm. B7, p. 345].

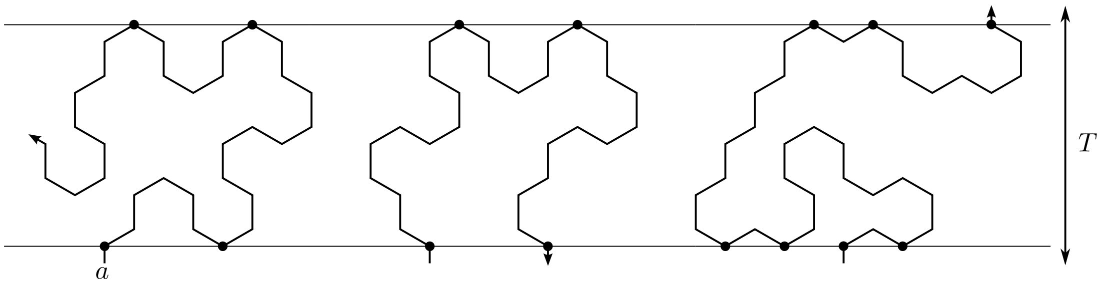

FIGURE 8. Walks confined to a strip of height $T = 5$ with weights attached to vertices along the top and bottom of the strip: a general walk, an arch, and a bridge.

Proof. Again, the existence of the limits follows from concatenation and unfolding arguments as given in Section 4 of [16]. The log-convexity result is easily adapted from [16, Thm. 6.3]. ∎

Therefore the growth constant for interacting SAWs in a strip is independent of which wall the interacting monomers are situated on. As per our discussion in Section 2, it turns out to be convenient to put the interacting monomers on the top, rather than at the bottom.

The next proposition describes how the growth constant $\mu_T(1, y)$ changes as T grows.

Proposition 7. *For $y > 0$, we have*

$$\mu_T(1, y) < \mu_{T+1}(1, y).$$

Moreover, as $T \to \infty$,

$$\mu_T(1, y) \to \mu(y),$$

the growth constant of SAWs interacting with a surface (Proposition 5).

Proof. Again, the proof is an adaptation to the honeycomb lattice of results proved by van Rensburg, Orlandini and Whittington for the hypercubic lattices [16] (similar arguments are also covered in Chapter 8 of [20], but without interactions). Our arguments are similar to Sections 5 and 6 of [16], but, we believe, somewhat shorter[5].

First, since $\mu_T(1, y) = \mu_T(y, 1)$, we may choose to work with arches in a strip of height T, interacting with the bottom line of the strip. Let us say that an arch going from mid-edge a to mid-edge b is *unfolded* if the abscissa $\mathbf{x}(v)$ of every non-final vertex v of the walk satisfies $\mathbf{x}(a) \leq \mathbf{x}(v) < \mathbf{x}(b)$. That is, an arch is unfolded if its starting point is (not necessarily strictly) to the left of all other points, and its final point is strictly to the right of all other points. Two unfolded arches w_1 and w_2 can be concatenated (after deleting the last half-edge of the first arch and the first half-edge of the second arch, see Fig. 9) to form a new unfolded arch w. Observe that

$$|w| = |w_1| + |w_2| - 1 \quad \text{and} \quad c(w) = c(w_1) + c(w_2) - 1.$$

We say an unfolded arch is *prime* if it is not the concatenation of two (or more) unfolded arches. The first two arches of Fig. 9 are prime, the third one, by construction, is not.

Let us fix $y > 0$. The arguments of [16, Section 4] show that the generating function $\vec{A}_T(x, y)$ that counts unfolded arches (by the size and the number of contacts with the bottom line of the strip) has the same radius of convergence as the generating function $A_T(x, y)$ that counts

[5]In particular, working in two dimensions gives a simple argument proving the divergence at their radius of convergence of generating functions that count SAWs in a strip. Moreover, we do not need the full strength of a pattern theorem.

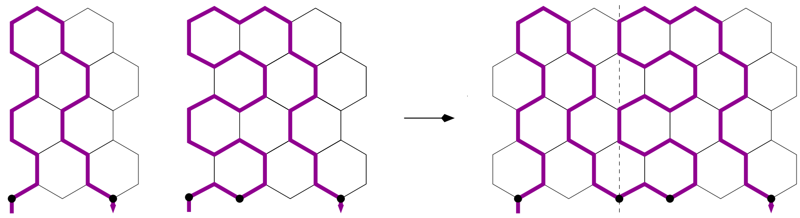

FIGURE 9. Concatenation of two unfolded arches in a strip of height $T = 5$.

all arches. By Proposition 6, this radius is $\rho_T(y) := 1/\mu_T(y)$. Moreover, the above definition of prime arches shows that

$$\vec{A}_T(x,y) = \frac{P_T(x,y)}{1 - P_T(x,y)/(xy)},$$

where $P_T(x,y)$ counts prime unfolded arches.

It follows from the transfer matrix method that the series $\vec{A}_T(x,y)$ (and, in fact, all series counting walks in a strip that occur in this section) is a rational function of x and y (see [13, p. 364], or [1]). Hence $\vec{A}_T(x,y)$ diverges at its radius $\rho_T(y)$, and it follows that $P_T(\rho_T(y), y)/(y\rho_T(y)) = 1$.

Now consider the prime unfolded arch w that consists of a (wavy) column with $2(T-1)$ vertical edges (like the first arch of Fig. 9). This walk contributes a term $x^{4T-1}y^2$ in the series $P_T(x,y)$. Let $\tilde{P}_T(x,y) := P_T(x,y) - x^{4T-1}y^2$. The generating function of unfolded arches that do not contain w as a factor is

$$\frac{\tilde{P}_T(x,y)}{1 - \tilde{P}_T(x,y)/(xy)}.$$

Its radius is reached at the point x satisfying $\tilde{P}_T(x,y)/(xy) = 1$. It is hence larger than the radius $\rho_T(y)$ of $\vec{A}_T(x,y)$. The above series counts (among others) walks that do not touch the top line of the strip. Their generating function is $\vec{A}_{T-1}(x,y)$, which has radius $\rho_{T-1}(y)$. Hence $\rho_{T-1}(y) > \rho_T(y)$, or equivalently $\mu_{T-1}(y) < \mu_T(y)$.

The proof that $\mu_T(y)$ tends to $\mu(y)$ is analogous to the proof of Theorem 6.5 in [16].

■

We now derive a corollary that will be essential in the next section. It deals with the properties of $\rho_T(y) := 1/\mu_T(1,y)$, which is the radius of convergence of the series

$$C_T(x,y) := \sum_{k\geq 0} c_{T,k}(1,y)x^k$$

counting walks in a strip that interact with the top boundary, and of the analogous series $A_T(x,y)$ and $B_T(x,y)$ that count arches and bridges. See Fig. 10 for an illustration.

Corollary 8. *Let $y > 0$. The generating functions $A_T(x,y)$, $B_T(x,y)$ and $C_T(x,y)$ all have the same radius of convergence,*

$$\rho_T(y) = 1/\mu_T(1,y).$$

Moreover, $\rho_T(y)$ decreases to $\rho(y) := 1/\mu(y)$ as T goes to infinity. In particular, $\rho_T(y)$ decreases to $1/\mu$ for $y \leq y_c$.

There exists a unique $y_T > 0$ such that $\rho_T(y_T) = x_c = 1/\mu$. The series (in y) $A_T(x_c, y)$, $B_T(x_c, y)$ and $C_T(x_c, y)$ have radius of convergence y_T, and y_T decreases to the critical fugacity y_c as T goes to infinity.

Proof. The first part of the corollary is an obvious translation of Propositions 6 and 7.

The existence of y_T follows from the intermediate value theorem: ρ_T is continuous, $\rho_T(1) > \rho(1) = x_c$ and $\rho_T(y) \to 0$ as $y \to \infty$ (because $\rho_T(y) \leq 1/\sqrt{y}$ as can be seen by counting zig-zag paths, as in the proof of Proposition 5).

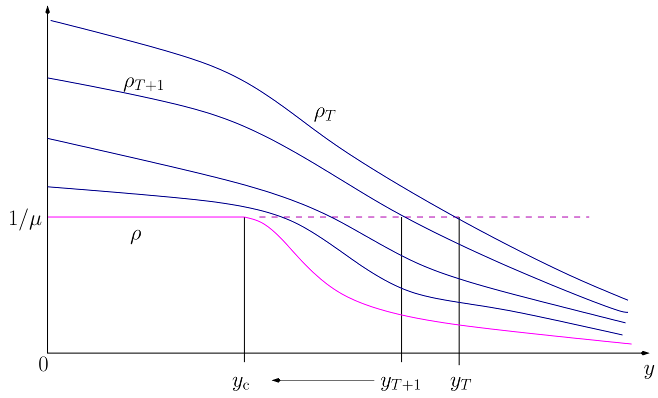

FIGURE 10. An illustration of Corollary 8.

The uniqueness of y_T follows from the log-convexity of $\mu_T(y)$ in $\log y$, together with $\rho_T(1) > x_c$: this precludes having $\rho_T(y) = \rho_T(y') = x_c$ with $y \neq y'$. This also means that

$$\rho_T(y) < \rho_T(y_T) \Longleftrightarrow y > y_T \qquad \text{and} \qquad \rho_T(y) > \rho_T(y_T) \Longleftrightarrow y < y_T. \tag{15}$$

Let us now prove that y_T is the radius of convergence of $A_T(x_c, y)$, $B_T(x_c, y)$ and $C_T(x_c, y)$. The argument is the same for the three series, so let us work for instance with C_T. We must first explain why $C_T(x_c, y)$ is indeed a series in y, that is, why the length generating function of SAWs in the T-strip having a fixed number of top contacts is finite at x_c. The reason for that is that the number of k-step walks of this type grows (at most) like $\mu_T(1,1)^k$, by definition of μ_T, and that $\mu_T(1,1) < 1/x_c$ as stated at the beginning of the corollary. Hence $C_T(x_c, y)$ is indeed a series in y. Now by definition of ρ_T, the series $C_T(x_c, y)$ converges if $x_c < \rho_T(y)$, and diverges if $x_c > \rho_T(y)$. But $x_c = \rho_T(y_T)$, so by (15), this means that $C_T(x_c, y)$ converges if $y < y_T$ and diverges if $y > y_T$, which means that y_T is the radius of convergence of $C_T(x_c, y)$.

Let us finally prove that y_T decreases towards y_c. First, since $\rho_T(y_T) = x_c$ and $\rho_{T+1}(y) < \rho_T(y)$ (Proposition 7), we have $\rho_{T+1}(y_T) < x_c$ and thus $y_{T+1} < y_T$. Hence the sequence $(y_T)_{T\geq 1}$ decreases. Let $\bar{y}$ be its limit. For $y \leq y_c$, we have $\rho_T(y) > \rho(y) = x_c$, and thus $y_T > y_c$ for all T. Hence $\bar{y} \geq y_c$. Since $\bar{y} < y_T$, we have $\rho_T(\bar{y}) > \rho_T(y_T) = x_c$, and thus $\rho(\bar{y}) \geq x_c$ (Proposition 7). Since $\rho(y) < x_c$ for $y > y_c$ (Proposition 6), it follows that $\bar{y} \leq y_c$. We have thus proved that y_T decreases to y_c. ■

4. THE CRITICAL SURFACE FUGACITY OF SAWS IS $1 + \sqrt{2}$

4.1. THE GLOBAL IDENTITY

Let us consider the identity (11) at $n = 0$, that is, at $\theta = \pi/2$. Then no loops are allowed. In particular, the polynomial $A^\circ_{T,L}$ reduces to 1. We focus now on the dilute regime (5), where

$$x_c^{-1} = 2\cos\left(\frac{\pi}{8}\right) = \sqrt{2+\sqrt{2}} \qquad \text{and} \qquad y^* = 1 + \sqrt{2}.$$

The global identity reads

$$1 = \alpha A_{T,L}(x_c, y) + \varepsilon E_{T,L}(x_c, y) + \beta(y) B_{T,L}(x_c, y), \tag{16}$$

where

$$\alpha = \cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}, \qquad \varepsilon = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \qquad \beta(y) = \frac{y^* - y}{y(y^*-1)} = \frac{1+\sqrt{2}-y}{\sqrt{2}\,y}.$$

4.2. A lower bound on y_c

Let us fix T, and set $y = y^*$ in (16). Since $\beta(y^*) = 0$, we obtain:

$$1 = \alpha A_{T,L}(x_c, y^*) + \varepsilon E_{T,L}(x_c, y^*).$$

As L increases, the numbers $A_{T,L}(x_c, y^*)$ count more and more walks. Hence they increase with L. Since the coefficients α and ε are positive, the above identity shows that $A_{T,L}(x_c, y^*)$ remains bounded as L increases. Hence the limit

$$\lim_{L\to\infty} A_{T,L}(x_c, y^*)$$

exists and is finite. Clearly, this limit is $A_T(x_c, y^*)$ where $A_T(x, y)$ is the generating function of arches in a T-strip, defined just above Corollary 8. According to this corollary, $A_T(x_c, y)$ has radius y_T. Since it converges at y^*, this means that $y^* \le y_T$. Since $y_T \to y_c$, we thus have

$$y^* \le y_c. \tag{17}$$

4.3. A limit identity

Proposition 9. *For $0 \le y < y_T$ (the radius of convergence of $A_T(x_c, \cdot)$ and $B_T(x_c, \cdot)$), the series counting arches and bridges in a T-strip satisfy*

$$\alpha A_T(x_c, y) + \beta(y) B_T(x_c, y) = 1. \tag{18}$$

Proof. Let us first prove that

$$\lim_L E_{T,L}(x_c, y) = 0 \quad \text{for } 0 \le y < y_T.$$

Indeed, $E_{T,L}(x_c, y)$ counts some self-avoiding walks of length at least L, starting from a, and confined to a T-strip. But the generating function of walks in the T-strip converges at (x_c, y) for $y < y_T$ (see Corollary 8), and thus its remainder of order L tends to 0 as L grows. This remainder is an upper bound on $E_{T,L}(x_c, y)$, which thus tends to 0 as well.

Taking the limit of (16) as $L \to \infty$ gives the proposition. ■

4.4. Convergence of $B_T(x_c, 1)$ to 0

This is a key point in our argument, and also a result of independent interest.

Theorem 10. *The length generating function $B_T(x, 1)$ counting bridges in a strip of height T, taken at the critical value $x_c = 1/\sqrt{2 + \sqrt{2}}$, tends to 0 as T tends to infinity.*

The proof, of a probabilistic nature, is given in the appendix. Let us note that the fact that $B_T(x_c, 1)$ converges (and actually decreases) follows easily from the case $y = 1$ of (18). Indeed, $A_T(x_c, 1)$ increases with T, but remains bounded since α and $\beta(1)$ are positive. Thus $A_T(x_c, 1)$ has a finite limit when T increases, and this limit is the generating function $A(x_c)$ counting arches in a half-plane. It then follows from (18) that $B_T(x_c, 1)$ decreases as T grows, and

$$\lim_T B_T(x_c, 1) = 1 - \alpha A(x_c). \tag{19}$$

Theorem 10 thus implies that $A(x_c) = 1/\alpha$.

Remarks

1. We can actually prove that $A_T(x_c, y) \to A(x_c)$ for $y < y^*$, but this will not be needed here. Returning to (18), this implies that $B_T(x_c, y) \to 0$ for $0 \le y < y^*$.

2. As discussed in [12, Remark 2], it follows from the SLE predictions of [19, Sec. 3.3.3 and 3.4.3] that $B_T(x_c, 1)$ is expected to decay as $T^{-1/4}$ as $T \to \infty$.

4.5. An upper bound on y_c

The series $A_{T+1}(x_c, y)$ counts arches of height at most $T+1$. This includes arches of height at most T, which have no contacts with the top boundary. Such arches are counted by $A_T(x_c, 1)$. Now consider an arch that has contacts with the boundary. By looking at its last contact, one can factor the arch into two bridges (see Fig. 11), and thus obtain

$$A_{T+1}(x_c, y) - A_T(x_c, 1) \le x_c B_T(x_c, 1) B_{T+1}(x_c, y). \tag{20}$$

This inequality holds in the domain of convergence of the series it involves, that is, for $y < y_{T+1}$, and thus in particular at y_c. Let us now write (18), first for $T+1$ and $y = y_c$ and then for T and $y = 1$:

$$\alpha A_{T+1}(x_c, y_c) + \beta(y_c) B_{T+1}(x_c, y_c) = 1 = \alpha A_T(x_c, 1) + B_T(x_c, 1).$$

Combine this with the inequality (20), taken at $y = y_c$. This gives

$$B_T(x_c, 1) - \beta(y_c) B_{T+1}(x_c, y_c) \le \alpha x_c B_T(x_c, 1) B_{T+1}(x_c, y_c),$$

or equivalently,

$$0 \le \frac{1}{B_{T+1}(x_c, y_c)} \le \alpha x_c + \frac{1}{B_T(x_c, 1)} \frac{y^* - y_c}{y_c(y^* - 1)}.$$

Recall that $B_T(x_c, 1)$ tends to 0 (Theorem 10). This forces $y^* \ge y_c$, otherwise the right-hand side would become arbitrarily large in modulus and negative as $T \to \infty$.

Together with (17), this establishes $y_c = y^* = 1 + \sqrt{2}$ and completes the proof of Theorem 2.

■

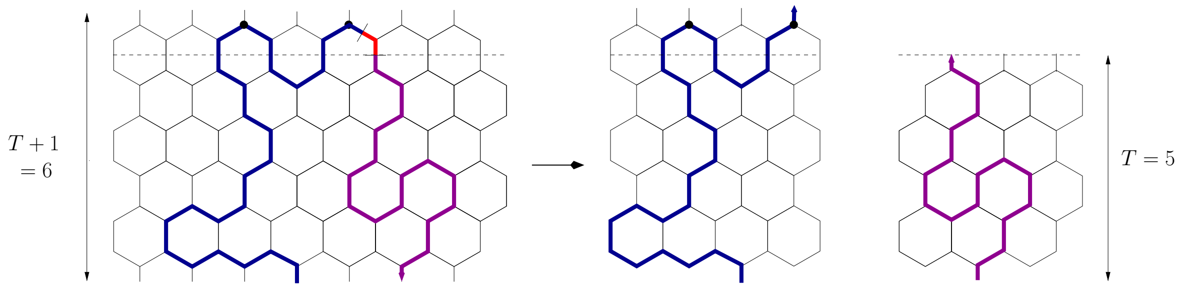

FIGURE 11. Factorisation of an arch of height $T+1$ into two bridges, of height $T+1$ and T respectively.

Acknowledgments

We thank Neal Madras, Andrew Rechnitzer, Stu Whittington and Alain Yger for helpful conversations. AJG and JdG acknowledge financial support from the Australian Research Council. NRB was supported by the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS). HDC was supported by the ANR grant BLAN06-3-134462, the ERC AG CONFRA, as well as by the Swiss FNS.

Part of this work was carried out during visits of the authors to the Mathematical Sciences Research Institute in Berkeley, during the Spring 2012 Random Spatial Processes Program. The authors thank the institute for its hospitality and the NSF (grant DMS-0932078) for its financial support.

◁ ◁ ◇ ▷ ▷

Appendix. Proof of Theorem 10.

Before starting the proof, let us introduce some additional notation. The set of mid-edges of the honeycomb lattice is denoted by $\mathbb{H}$. The lattice has an origin $a \in \mathbb{H}$, at coordinates $(0, 0)$. We denote by $(\mathbf{x}(v), \mathbf{y}(v))$ the coordinates of a point $v \in \mathbb{C}$ (that is, its real and imaginary parts). We consider self-avoiding walks that start and end at a mid-edge. A self-avoiding walk γ is denoted by the sequence $(\gamma_0, \dots, \gamma_n)$ of its mid-edges. The length of γ, that is, the number of vertices of the lattice it visits, is denoted as before by $|\gamma| = n$. (This n has nothing to do with the n of the $O(n)$ model considered in Sections 1 and 2. We are dealing in this appendix with SAWs, that is, with the $O(0)$ model.) To lighten notation, we often omit floor symbols, especially in indices: for instance, γ_t should be understood as $\gamma_{\lfloor t \rfloor}$. The cardinality of a set A is denoted by $|A|$.

We have so far discussed bridges in a strip of height T (Fig. 8, right), which we call bridges *of height* T. In general, we call *bridge* any self-avoiding walk $\gamma = (\gamma_0, \dots, \gamma_n)$ that is a bridge of height T for some T. Equivalently, $\mathbf{y}(\gamma_0) < \mathbf{y}(\gamma_i) < \mathbf{y}(\gamma_n)$ for $0 < i < n$. The set of bridges of length n is denoted by SAB_n.

The set R_γ of *renewal points* of $\gamma \in \mathrm{SAB}_n$ is the set of points of the form γ_i with $0 \le i \le n$, for which $\gamma_{[0,i]} := (\gamma_0, \dots, \gamma_i)$ and $\gamma_{[i,n]} := (\gamma_i, \dots, \gamma_n)$ are bridges. We denote by $\mathbf{r}_0(\gamma), \mathbf{r}_1(\gamma), \dots$ the indices of the renewal points. That is, $\mathbf{r}_0(\gamma) = 0$ and $\mathbf{r}_{k+1}(\gamma) = \inf\{j > \mathbf{r}_k(\gamma) : \gamma_j \in \mathsf{R}_\gamma\}$ for each k. When no confusion is possible, we often denote $\mathbf{r}_k(\gamma)$ by $\mathbf{r}_k$.

A bridge $\gamma \in \mathrm{SAB}_n$ is *irreducible* if its only renewal points are γ_0 and γ_n. Let iSAB be the set of irreducible bridges of arbitrary length starting from a. Every bridge γ is the concatenation of a finite number of irreducible bridges, the decomposition is unique and the set R_γ is the union of the initial and terminal points of the bridges that comprise this decomposition.

Kesten's relation for irreducible bridges (see [20, Section 4.2] or [17]) on the hypercubic lattice $\mathbb{Z}^d$ can be easily adapted to the honeycomb lattice. It gives

$$\sum_{\gamma \in \mathrm{iSAB}} x_{\mathrm{c}}^{|\gamma|} = 1.$$

This enables us to define a probability measure $\mathbb{P}_{\mathrm{iSAB}}$ on iSAB by setting $\mathbb{P}_{\mathrm{iSAB}}(\gamma) = x_{\mathrm{c}}^{|\gamma|}$. Let $\mathbb{P}_{\mathrm{iSAB}}^{\otimes \mathbb{N}}$ denote the law on semi-infinite walks $\gamma : \mathbb{N} \to \mathbb{H}$ formed by the concatenation of infinitely many independent samples $\gamma^{[1]}, \gamma^{[2]}, \dots$ of $\mathbb{P}_{\mathrm{iSAB}}$. We refer to [20, Section 8.3] for details of related measures in the case of $\mathbb{Z}^d$. The definition of R_γ and the indexing of renewal points extend to this context (we obtain an infinite sequence $(\mathbf{r}_k)_{k \in \mathbb{N}}$).

Note that a bridge of length 2 has height 1 and ends at ordinate $3/2$ (since edges have unit length). More generally, a bridge γ of length n has height $\mathsf{H}(\gamma) = \frac{2}{3}\mathbf{y}(\gamma_n)$. We define the height of a general SAW γ similarly. The *width* of γ is defined by

$$\mathsf{W}(\gamma) = \frac{1}{\sqrt{3}} \max\{\mathbf{x}(\gamma_k) - \mathbf{x}(\gamma_{k'}), 0 \le k, k' \le n\},$$

so that a bridge of length 2 has width $1/2$.

We have proved in Section 4.4 that $B_T(x_{\mathrm{c}}, 1)$ converges as $T \to \infty$. We provide here an alternative proof, and relate the limiting value to the average height of irreducible bridges.

Lemma 11. *As* $T \to \infty$,

$$B_T(x_{\mathrm{c}}, 1) \to \frac{1}{\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))}.$$

Proof. The result follows from standard renewal theory. We can for instance apply [20, Theorem 4.2.2(b)] to the sequence

$$f_T := \sum_{\gamma \in \mathrm{iSAB}:\ \mathsf{H}(\gamma) = T} x_{\mathrm{c}}^{|\gamma|}.$$

Indeed, with the notation of this theorem, $v_T = B_T(x_{\mathrm{c}}, 1)$ and $\sum_k k f_k = \mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))$. ∎

Thus Theorem 10 is equivalent to

$$\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) = \infty.$$

We will prove this by contradiction. Assuming $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$ is finite, we first show that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma))$ is also finite. Then, we show that under these two conditions, an infinite bridge is very narrow. The last step consists in proving that this cannot be the case. The argument uses a *stickbreak* operation which perturbs a bridge by selecting a subpath and rotating it clockwise by $\frac{\pi}{3}$. The new path is a self-avoiding bridge for an adequately chosen subpath. But its width is relatively large, contradicting the fact that bridges are narrow. The strategy of proof is greatly inspired by a recent paper of Duminil-Copin and Hammond, where self-avoiding walks are proved to be sub-ballistic [11]. The additional difficulty here comes from the fact that Section 4 of [11] (which corresponds to the proof presented here) relies on the assumption $\mathbb{E}_{\text{iSAB}}(|\gamma|) < \infty$, which is stronger than the assumption $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$ that we have here. In particular, we need the following result.

Proposition 12. *If* $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$, *then* $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) < \infty$.

To prove this proposition we will first establish some simple lemmas regarding SAW generating functions in a slightly different geometry to that used in Section 2.

Consider the rectangular domain $R_{T,L}$ depicted in Fig. 12, with its boundary partitioned into four subsets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}^-$ and $\mathcal{E}^+$ (the mid-edges of $\mathcal{E}^+$ point up, those of $\mathcal{E}^-$ point down, on both sides of the rectangle). We do not consider any interactions here. As in Section 2, we define four generating functions counting self-avoiding walks in the rectangle, going from a to a mid-edge of the boundary. First, we set

$$\tilde{A}_{T,L}(x) := \sum_{\gamma: a \rightsquigarrow \mathcal{A}\setminus\{a\}} x^{|\gamma|},$$

and then the generating functions $\tilde{B}_{T,L}(x)$, $\tilde{E}^-_{T,L}(x)$ and $\tilde{E}^+_{T,L}(x)$ are defined similarly. We then have the following lemma, which can be viewed as a translation of (16) to the new rectangular domain $R_{T,L}$. We omit the proof, as it is essentially the same as that of Proposition 4 with $n = 0$ and $y = 1$ in the dilute regime.

Lemma 13. *The generating functions* $\tilde{A}_{T,L}$, $\tilde{B}_{T,L}$, $\tilde{E}^+_{T,L}$ *and* $\tilde{E}^-_{T,L}$, *evaluated at* $x = x_c$, *satisfy the identity*

$$1 = \alpha \tilde{A}_{T,L}(x_c) + \tilde{B}_{T,L}(x_c) + \varepsilon^+ \tilde{E}^+_{T,L}(x_c) + \varepsilon^- \tilde{E}^-_{T,L}(x_c),$$

where, as before, $\alpha = \cos(\frac{3\pi}{8})$, *and now* $\varepsilon^- = \cos(\frac{\pi}{4})$ *and* $\varepsilon^+ = \cos(\frac{\pi}{8})$.

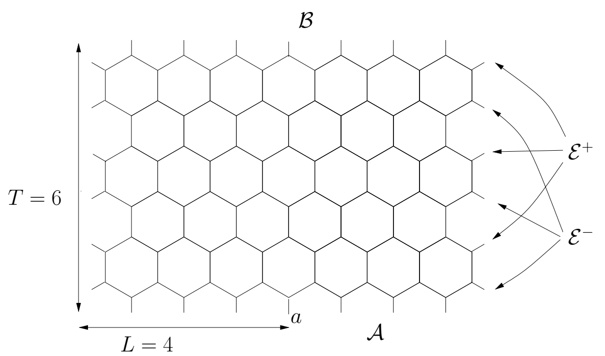

FIGURE 12. The rectangular domain $R_{T,L}$ with $T = 6$ and $L = 4$.

Convention. Since we always evaluate our generating functions at $x = x_c$, we will almost systematically omit the variable x_c, so that $\tilde{A}_{T,L}$ now means $\tilde{A}_{T,L}(x_c)$, and so on.

We now wish to take the size of the rectangle $R_{T,L}$ to infinity to obtain a half-plane, as we did in the previous geometry. This time, however, we want T and L to increase together, so the situation here is a bit more delicate.

Recall that an *arch* is a self-avoiding walk starting from a, confined to the upper half-plane, and ending on the line $\mathbf{y} = 0$. As proved in Section 4.4, the generating function $A(x)$ of arches converges at x_c. Hence, if $\mathfrak{a}_L(x)$ denotes the generating function of arches ending L cells to the right of the initial mid-edge a, the sequence $\mathfrak{a}_L \equiv \mathfrak{a}_L(x_c)$ tends to 0 as $L \to \infty$.

Lemma 14. *Assume that $T \equiv T_k$ and $L \equiv L_k$ both tend to infinity as k grows, and that $T\mathfrak{a}_{2L} \longrightarrow 0$. If $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$, then*

$$\lim_{k\to\infty} \tilde{B}_{T_k,L_k} > 0.$$

Proof. We begin by bounding $\tilde{E}^{\pm}_{T,L}$ in terms of $\mathfrak{a}_{2L}$. For $m \in \mathbb{N}$, let $\mathfrak{e}^+_m(x)$ be the generating function of walks in $R_{T,L}$ ending on the right side of the rectangle, on the mth row of $\mathcal{E}^+$, so that, by symmetry, $\tilde{E}^+_{T,L} = 2\sum_{m\le\lfloor\frac{T}{2}\rfloor} \mathfrak{e}^+_m$. The Cauchy-Schwarz inequality gives

$$\left(\tilde{E}^+_{T,L}\right)^2 \le 4\lfloor\tfrac{T}{2}\rfloor \sum_{m\le\lfloor\frac{T}{2}\rfloor} (\mathfrak{e}^+_m)^2.$$

Now one can concatenate two walks contributing to $\mathfrak{e}^+_m$ (after reflecting the second one) by adding a step between them in order to create an arch contributing to $\mathfrak{a}_{2L}$. This gives

$$\left(\tilde{E}^+_{T,L}\right)^2 \le 4\lfloor\tfrac{T}{2}\rfloor\, x_c^{-1}\mathfrak{a}_{2L}.$$

We obtain a similar upper bound for $\tilde{E}^-_{T,L}$ with $\lfloor\frac{T}{2}\rfloor$ replaced by $\lceil\frac{T}{2}\rceil$.

The assumptions of the lemma now imply that $\tilde{E}^+_{T,L}$ and $\tilde{E}^-_{T,L}$ tend to 0. Moreover, $\tilde{A}_{T,L}$ increases with L and T, and converges to $A \equiv A(x_c)$. Returning to Lemma 13 shows that $\tilde{B}_{T,L}$ must also converge, and gives

$$\begin{aligned}\lim_k \tilde{B}_{T_k,L_k} &= 1 - \alpha A(x_c)\\ &= \lim_T B_T(x_c, 1) \qquad \text{by (19)}\\ &> 0 \quad \text{by assumption.}\end{aligned}$$

∎

Proof of Proposition 12. Let us now return to random infinite bridges and use them to give an upper bound on $\tilde{B}_{T,L}$. Let $0 < \delta < 1/\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma))$. We have

$$\begin{aligned}\tilde{B}_{T,L} &= \sum_{\gamma:a\rightsquigarrow\mathcal{B}} x_c^{|\gamma|}\\ &\le \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\exists n \in \mathbb{N} : \mathsf{H}(\gamma_{[0,\mathbf{r}_n]}) = T \text{ and } \mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \le 2L\big)\\ &\le \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\exists n \ge \delta T : \mathsf{H}(\gamma_{[0,\mathbf{r}_n]}) = T \text{ and } \mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \le 2L\big).\end{aligned}$$

Let $\gamma^{[i]}$ be the i^{th} irreducible bridge of γ. Since the $\gamma^{[i]}$s are independent, we obtain

$$\begin{aligned}\tilde{B}_{T,L} &\le \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\forall i \le \delta T, \mathsf{W}(\gamma^{[i]}) \le 2L\big)\\ &= \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) \le 2L)^{\delta T}\\ &\le \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \ge T\big) + \exp\left(-\delta T\, \mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)\right). \qquad (21)\end{aligned}$$

Note that

$$\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) = \sum_{i=1}^{\delta T} \mathsf{H}(\gamma^{[i]}).$$

Hence the law of large numbers, together with the fact that $\delta \cdot \mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < 1$, implies that $\mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{H}(\gamma_{[0,\mathbf{r}_{\delta T}]}) \geq T\big)$ tends to 0 as $T \to \infty$. Hence, if $T \equiv T_k$ and $L \equiv L_k$ are such that T and $T\,\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)$ both tend to infinity, then $\tilde{B}_{T,L}$ tends to zero.

We now argue *ad absurdum.* Assume that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) = \infty$. Then

$$\limsup_{L\to\infty} \frac{\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L)}{\mathfrak{a}_{2L}} = \infty,$$

since $\mathfrak{a}_L$ is the term of a converging series (namely, the generating function $A(x_c)$ of arches) and $\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > L)$ is non-increasing in L and is the term of a diverging series (indeed, it sums to $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) = \infty$). Let $(L_k)_k$ be a sequence such that

$$\lim_{k\to\infty} \frac{\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L_k)}{\mathfrak{a}_{2L_k}} = \infty,$$

and take

$$T_k = \left\lfloor \frac{1}{\sqrt{\mathfrak{a}_{2L_k}\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L_k)}} \right\rfloor.$$

Then

$$T_k\,\mathbb{P}_{\text{iSAB}}(\mathsf{W}(\gamma) > 2L_k) \to \infty \quad \text{and} \quad T_k\,\mathfrak{a}_{2L_k} \to 0.$$

It follows from (21) that $\lim_k \tilde{B}_{T_k,L_k} = 0$. But T_k and L_k also satisfy Lemma 14, so we also have $\lim_k \tilde{B}_{T_k,L_k} > 0$, a contradiction. We thus conclude that $\mathbb{E}_{\text{iSAB}}(\mathsf{W}(\gamma)) < \infty$. ∎

Let Ω be the set of bi-infinite walks $\gamma : \mathbb{Z} \to \mathbb{H}$ such that $\gamma_0 = a$. Let $(\gamma^{[i]}, i \in \mathbb{Z})$ be a bi-infinite sequence of irreducible bridges sampled independently according to $\mathbb{P}_{\text{iSAB}}$. Let $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}$ denote the law on Ω formed by concatenating the bridges $\gamma^{[i]}, i \in \mathbb{Z}$ in such a way that $\gamma^{[1]}$ starts at a. Let $\mathcal{F}$ be the σ-algebra generated by events depending on a finite number of vertices of the walk.

We extend the indexing of renewal points to the bi-infinite walks of Ω. If $\gamma \in \Omega$ is a bi-infinite bridge (which is the case with probability 1 under $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}$), we obtain a bi-infinite sequence $(\mathbf{r}_n(\gamma))_{n\in\mathbb{Z}}$ such that $r_0(\gamma) = 0$. Let $\tau : \Omega \to \Omega$ be the *shift* defined by $\tau(\gamma)_i = \gamma_{i+\mathbf{r}_1(\gamma)} - \gamma_{\mathbf{r}_1(\gamma)}$ for every $i \in \mathbb{Z}$. The shift translates the walk so that $\mathbf{r}_1(\gamma)$ is now at the origin a of the lattice. Note that $\mathbf{r}_i(\tau(\gamma)) = \mathbf{r}_{i+1}(\gamma) - \mathbf{r}_1(\gamma)$. Let σ denote the reflection in the real axis.

Proposition 15. *The measure $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}$ satisfies the following properties.*

- (P_1) *It is invariant under the shift τ.*
- (P_2) *The shift τ is ergodic for $(\Omega, \mathcal{F}, \mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}})$.*
- (P_3) *Under $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}$, the random variables $(\sigma\gamma_n)_{n\leq 0}$ and $(\gamma_n)_{n\leq 0}$ are independent and identically distributed.*

Proof. Property (P_1) is fairly straightforward. Indeed, for every $n > 0$, the law of $\gamma_{[\mathbf{r}_{-n}(\gamma),\mathbf{r}_n(\gamma)]}$ determines, in the high-n limit, the law of γ (since we work with the σ-algebra $\mathcal{F}$). Now, the laws of $\tau(\gamma_{[\mathbf{r}_{-n+1}(\gamma),\mathbf{r}_{n+1}(\gamma)]})$ and $\gamma_{[\mathbf{r}_{-n}(\gamma),\mathbf{r}_n(\gamma)]}$ are the same by construction (both are the law of $2n$ concatenated independent irreducible bridges). Thus (P_1) follows by letting $n \to \infty$.

Let us turn to (P_2). Consider a shift-invariant event $\mathfrak{A}$. We want to show that $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}(\mathfrak{A}) \in \{0,1\}$. Let $\varepsilon > 0$. There exists $n > 0$ and an event $\mathfrak{A}_n$ depending only on the vertices $\gamma_{-n}, \ldots, \gamma_n$ such that $\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}(\mathfrak{A}_n \,\Delta\, \mathfrak{A}) \leq \varepsilon$, where Δ denotes the symmetric difference. In particular, $|\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}(\mathfrak{A}) - \mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}(\mathfrak{A}_n)| \leq \varepsilon$. By extension, $\mathfrak{A}_n$ depends only on vertices in $\gamma_{\mathbf{r}_{-n}}, \ldots, \gamma_{\mathbf{r}_n}$. Invariance of $\mathfrak{A}$ under τ implies that $\mathfrak{A} = \tau^{-2n}(\mathfrak{A})$, so that

$$\mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}(\mathfrak{A}) = \mathbb{P}^{\otimes\mathbb{Z}}_{\text{iSAB}}\left(\mathfrak{A} \cap \tau^{-2n}(\mathfrak{A})\right). \tag{22}$$

Moreover,

$$\Big|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(\mathfrak{A}\cap\tau^{-2n}(\mathfrak{A})\big)-\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(\mathfrak{A}_n\cap\tau^{-2n}(\mathfrak{A}_n)\big)\Big|$$
$$\le \mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(\mathfrak{A}\,\Delta\,\mathfrak{A}_n\big)+\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}\big(\tau^{-2n}(\mathfrak{A})\,\Delta\,\tau^{-2n}(\mathfrak{A}_n)\big)\le 2\varepsilon\,.$$

Using (22) and the independence between the walk before and after $\mathbf{r}_n$, this reads

$$|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})-\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A}_n)^2|\le 2\varepsilon,$$

which, combined with $|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})-\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A}_n)|\le\varepsilon$, implies

$$|\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})-\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})^2|\le 4\varepsilon\,.$$

By letting ε tend to 0, we obtain that $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})=\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})^2$ and therefore $\mathbb{P}^{\otimes\mathbb{Z}}_{\mathrm{iSAB}}(\mathfrak{A})\in\{0,1\}$. Hence ($\mathrm{P}_2$) is proved.

Since the law of irreducible bridges is invariant (up to a translation) under reflection in a horizontal line, (P_3) is straightforward. ■

Renewal points separate a walk into two parts, located below and above the point. We now introduce a more restrictive notion, illustrated in Fig. 13 (left). A mid-edge γ_k of a walk γ is said to be a *diamond point* if

- it lies on a vertical edge of the lattice,
- the walk is contained in the cone

$$\big((\gamma_k-\tfrac{i}{2})+\mathbb{R}_+e^{i\pi/3}+\mathbb{R}_+e^{2i\pi/3}\big)\cup\big((\gamma_k+\tfrac{i}{2})-\mathbb{R}_+e^{i\pi/3}-\mathbb{R}_+e^{2i\pi/3}\big)$$

(recall that edges have length 1). The set of diamond points of γ is denoted by $\mathbf{D}_\gamma$. Of course, it is a subset of $\mathbf{R}_\gamma$. The following proposition tells us that, under our assumption, a positive fraction of renewal points of an infinite bridge are diamond points.

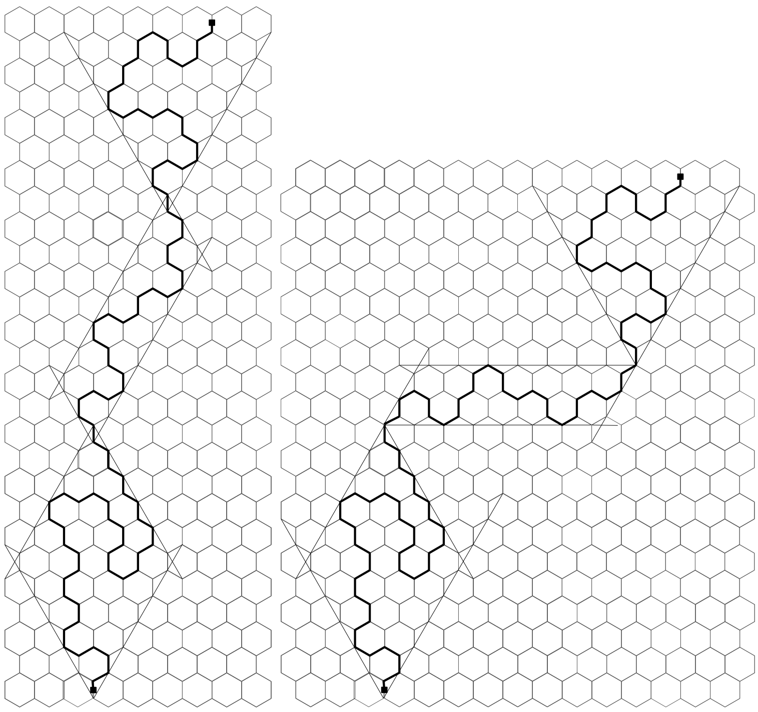

FIGURE 13. *Left*: A bridge having 3 diamond points. *Right*: A stickbreak operation applied to this bridge.

Proposition 16. *If* $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) < \infty$*, then there exists* $\delta > 0$ *such that*

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\left(\liminf_{n\to\infty} \frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n\}|}{n} \geq \delta\right) = 1.$$

Let us first provide a heuristic argument. Since $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))$ is finite, so is $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{W}(\gamma))$ (Proposition 12). Then $\mathbb{E}_{\mathrm{iSAB}}(\mathbf{x}(\gamma_{|\gamma|})) = 0$, and the law of large numbers implies that the prefixes of an infinite bridge are tall and skinny – that is, height grows linearly, width grows sub-linearly. So the probability of a bridge staying within a cone as thin as one likes is positive, and a similar result holds going backwards. Thus, diamond points occur with positive density among renewal points.

Proof. Let us first prove that $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{Z}}(\gamma_0 \in \mathbf{D}_\gamma) > 0$. Proposition 12 shows that $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{W}(\gamma)) < \infty$. Hence $\mathbb{E}_{\mathrm{iSAB}}(\mathbf{x}(\gamma_{|\gamma|}))$ is well-defined, and is 0 since the law of an irreducible bridge is invariant under reflection in the imaginary axis. The law of large numbers thus implies that, $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $\mathbf{x}(\gamma_{\mathbf{r}_n})/n \longrightarrow 0$. Since the expected width of irreducible bridges is finite, the law of large numbers shows that

$$\frac{1}{n}\sum_{i=1}^{n} \mathsf{W}(\gamma_{[\mathbf{r}_{i-1},\mathbf{r}_i]}) \longrightarrow c \quad \text{and} \quad \frac{1}{n+1}\sum_{i=1}^{n+1} \mathsf{W}(\gamma_{[\mathbf{r}_{i-1},\mathbf{r}_i]}) \longrightarrow c \qquad \text{a.s.},$$

where $c := \mathbb{E}_{\mathrm{iSAB}}(\mathsf{W}(\gamma))$ is a positive constant. The second identity reads

$$\left(1 - \frac{1}{n+1}\right)\frac{1}{n}\sum_{i=1}^{n} \mathsf{W}(\gamma_{[\mathbf{r}_{i-1},\mathbf{r}_i]}) + \frac{1}{n+1}\mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]}) \longrightarrow c < \infty \qquad \text{a.s.},$$

and comparing with the first identity shows that $\mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]})/n \longrightarrow 0$ almost surely. Thus

$$\frac{1}{n}\left(\frac{1}{\sqrt{3}}|\mathbf{x}(\gamma_{\mathbf{r}_n})| + \mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]})\right) \longrightarrow 0 \qquad \text{a.s.}$$

Since

$$\mathsf{W}(\gamma_{[0,\mathbf{r}_n]}) \leq 2\max\left\{\frac{1}{\sqrt{3}}|\mathbf{x}(\gamma_{\mathbf{r}_k})| + \mathsf{W}(\gamma_{[\mathbf{r}_k,\mathbf{r}_{k+1}]}),\ 0 \leq k \leq n-1\right\},$$

we find that, $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $\mathsf{W}(\gamma_{[0,\mathbf{r}_n]})/n \longrightarrow 0$.

Let us now apply the law of large numbers to $\mathbf{y}(\gamma_{\mathbf{r}_n})$. We obtain that, $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$-almost surely, $\mathbf{y}(\gamma_{\mathbf{r}_n})/n \longrightarrow \frac{3}{2}\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) > 0$.

Now define

$$I(\gamma) := \inf_{k\geq 0}\left(\mathbf{y}(\gamma_k) - \sqrt{3}|\mathbf{x}(\gamma_k)| + 1/2\right).$$

Note that for an infinite bridge $\gamma = (\gamma_0, \gamma_1, \ldots)$, the origin γ_0 is a diamond point if and only if $I(\gamma) \geq 0$.

By comparing a general point γ_k of γ with the last renewal point $\mathbf{r}_n$ before γ_k and the next one after γ_k, one finds

$$I(\gamma) \geq \inf_{n\geq 0}\left(\mathbf{y}(\gamma_{\mathbf{r}_n}) - \sqrt{3}\left(|\mathbf{x}(\gamma_{\mathbf{r}_n})| + \sqrt{3}\mathsf{W}(\gamma_{[\mathbf{r}_n,\mathbf{r}_{n+1}]})\right) + 1/2\right).$$

Then by the arguments presented earlier in this proof, it follows that $I(\gamma) > -\infty$ almost surely.

Let $K \in \mathbb{N}$ be such that $\rho_K := \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(I(\gamma) \geq -K) > 0$. We are going to show that

$$\rho_0 \geq (2x_c^4)^K \rho_K > 0. \tag{23}$$

To prove this, consider an experiment under which the law $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$ is constructed by first concatenating K independent samples of $\mathbb{P}_{\mathrm{iSAB}}$ (starting from a) and then an independent sample γ' of $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}$. If each of the K samples happens to be a walk of length 4 going from a to $a + 3i$ and $I(\gamma') \geq -K$, then the complete walk γ satisfies $I(\gamma) \geq 0$. The probability that the ith sample of $\mathbb{P}_{\mathrm{iSAB}}$ is a walk of length 4 going from a to $a + 3i$ is $2x_c^4$. Thus, the experiment behaves as described with probability $(2x_c^4)^K\rho_K$, and we obtain (23), that is, $\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(\gamma_0 \in \mathbf{D}_\gamma) > 0$.

Using Property (P_3) of Proposition 15, we deduce that

$$\delta := \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{Z}}(\gamma_0 \in \mathbf{D}_\gamma) = \left(\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(\gamma_0 \in \mathbf{D}_\gamma)\right)^2 > 0.$$

The shift τ being ergodic (cf. Property (P_2) of Proposition 15), the ergodic theorem, applied to $\mathbb{1}_{\gamma_0 \in \mathbf{D}_\gamma}$, gives

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{Z}}\left(\lim_{n\to\infty} \frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} = \delta\right) = 1\,.$$

Let γ be a bi-infinite bridge, and denote $\gamma^+ = \gamma_{[0,\infty)}$. Then for $n \geq 0$, $\mathbf{r}_n(\gamma) = \mathbf{r}_n(\gamma^+)$, and

$$\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\} = \mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma^+)\} \subset \mathbf{D}_{\gamma^+} \cap \{0,\ldots,\mathbf{r}_n(\gamma^+)\}$$

since all diamond points of γ are diamond points of γ^+. This implies that

$$\begin{aligned}\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}&\left(\liminf_{n\to\infty} \frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \geq \delta\right)\\ &= \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{Z}}\left(\liminf_{n\to\infty} \frac{|\mathbf{D}_{\gamma^+} \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \geq \delta\right)\\ &\geq \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{Z}}\left(\liminf_{n\to\infty} \frac{|\mathbf{D}_\gamma \cap \{0,\ldots,\mathbf{r}_n(\gamma)\}|}{n} \geq \delta\right) = 1.\end{aligned}$$

This concludes the proof of the proposition. ■

We now introduce some final definitions and a minor lemma before proving the main result. By Lemma 11, we want to prove that $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) = \infty$. We will argue *ad absurdum*. Henceforth, assume $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) < \infty$ and let $\nu > \mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma))$. Also, let $0 < \varepsilon < \delta/20$, where δ satisfies Proposition 16.

Let Ω^+ denote the set of semi-infinite walks in the upper half-plane. That is, $\phi = (\phi_0, \phi_1, \ldots) \in \Omega^+$ if and only if $\mathbf{y}(\phi_i) > 0$ for $i > 0$. For $\phi \in \Omega^+$ and γ a finite bridge, we write $\gamma \lhd \phi$ if $\phi_{[0,|\gamma|]} = \gamma$ and $\phi_{|\gamma|}$ is a renewal point of ϕ. Note that

$$x_c^{|\gamma|} = \mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}(\phi \in \Omega^+ : \gamma \lhd \phi)\,. \tag{24}$$

Let $\overline{\mathrm{SAB}}_n$ denote the set of finite bridges γ with exactly $n+1$ renewal points (meaning that $\mathbf{r}_n(\gamma) = |\gamma|$) such that

(C_1) $\mathsf{H}(\gamma) \leq \nu n$,

(C_2) $|\mathbf{D}_\gamma| \geq \delta n/2$.

Let us define $\overline{\mathrm{SAB}}_n^+ = \{\phi \in \Omega^+ : \exists \gamma \in \overline{\mathrm{SAB}}_n \text{ such that } \gamma \lhd \phi\}$. That is, the prefix of ϕ consisting of its n first irreducible bridges satisfies (C_1) and (C_2). It follows from (24) that

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big) = \sum_{\gamma \in \overline{\mathrm{SAB}}_n} x_c^{|\gamma|}\,. \tag{25}$$

Lemma 17. *Under the above assumptions, we have, as* $n \to \infty$,

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(\overline{\mathrm{SAB}}_n^+\big) \longrightarrow 1.$$

Proof. We consider Conditions (C_1) and (C_2) separately. Condition (C_1) for $\gamma \in \overline{\mathrm{SAB}}_n$ translates for $\phi \in \overline{\mathrm{SAB}}_n^+$ into $\mathsf{H}(\phi_{[0,\mathbf{r}_n]}) \leq \nu n$. Since $\mathbb{E}_{\mathrm{iSAB}}(\mathsf{H}(\gamma)) < \nu$, the law of large numbers gives

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\Big(\phi \in \Omega^+ : \mathsf{H}(\phi_{[0,\mathbf{r}_n]}) \leq \nu n\Big) \longrightarrow 1.$$

Let us now consider Condition (C_2), which translates into $|\mathbf{D}_{\phi_{[0,\mathbf{r}_n]}}| \geq \delta n/2$. But

$$\mathbf{D}_{\phi_{[0,\mathbf{r}_n]}} \supset \mathbf{D}_\phi \cap \{0,\ldots,,\mathbf{r}_n\},$$

since the truncation operation $\phi \to \phi_{[0,\mathbf{r}_n]}$ can only create (and not annihilate) diamond points. Thus Proposition 16 yields

$$\mathbb{P}_{\mathrm{iSAB}}^{\otimes\mathbb{N}}\big(|\mathbf{D}_{\phi_{[0,\mathbf{r}_n]}}| \geq \tfrac{\delta}{2} n\big) \longrightarrow 1,$$

and we have proved the lemma. ■

Proof of Theorem 10. We are going to prove that

$$\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\Big(\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]}) > \varepsilon n\Big) \geq \left(\frac{\delta n x_{\text{c}}}{10(\nu n+2)}\right)^2 \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\overline{\text{SAB}}_n^+\big)\,. \tag{26}$$

We proved at the beginning of the proof of Proposition 16 that $\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}(\phi)]})/n$ tends to zero almost surely under the $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}$-distribution. Thus the left-hand side of the above inequality tends to 0 as $n \to \infty$, and so does the right-hand side. This contradicts Lemma 17 and proves that our assumption $\mathbb{E}_{\text{iSAB}}(\mathsf{H}(\gamma)) < \infty$ cannot hold.

Consider $\gamma \in \overline{\text{SAB}}_n$. Let $\mathbf{d}_i$ be the index of the ith diamond point of γ. For integers $i \in \big[\frac{\delta}{10}n, \frac{2\delta}{10}n\big]$ and $j \in \big[\frac{3\delta}{10}n, \frac{4\delta}{10}n\big]$, let $\mathsf{StickBreak}_{i,j}(\gamma)$ be the following walk (see Fig. 13, right):

$$\mathsf{StickBreak}_{i,j}(\gamma) = \gamma_{[0,\mathbf{d}_i]} \circ s \circ \rho\big(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}\big) \circ \tilde{s} \circ \gamma_{[\mathbf{d}_j,\mathbf{r}_n]}\,, \tag{27}$$

where $\circ$ stands for the concatenation of walks, ρ is the clockwise rotation of angle $\pi/3$, s is a single right turn, and $\tilde{s}$ is a single left turn. The definition of diamond points implies that $\mathsf{StickBreak}_{i,j}(\gamma)$ is not only self-avoiding, but also a bridge. Also, note that we used (C_2) in order to define $\mathsf{StickBreak}(\gamma)$ for all these values of i and j.

Let

$$\Phi \;=\; \big[\tfrac{\delta}{10}n, \tfrac{2\delta}{10}n\big] \;\times\; \big[\tfrac{3\delta}{10}n, \tfrac{4\delta}{10}n\big] \;\times\; \overline{\text{SAB}}_n\,,$$

and denote

$$S := \sum_{(i,j,\gamma)\in\Phi} x_{\text{c}}^{|\mathsf{StickBreak}_{i,j}(\gamma)|}\,.$$

One can express S in terms of $\mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\overline{\text{SAB}}_n^+\big)$. Indeed, $|\mathsf{StickBreak}_{i,j}(\gamma)| = |\gamma| + 2$, and therefore

$$S = \sum_{(i,j,\gamma)\in\Phi} x_{\text{c}}^{|\gamma|+2} = \left(\frac{\delta n x_{\text{c}}}{10}\right)^2 \sum_{\gamma\in\overline{\text{SAB}}_n} x_{\text{c}}^{|\gamma|} = \left(\frac{\delta n x_{\text{c}}}{10}\right)^2 \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\overline{\text{SAB}}_n^+\big)\,. \tag{28}$$

We used (25) for the last equality. We are now going to give an upper bound on S, which will imply (26).

Note that the walk $\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}$ contains at least $\delta n/10$ diamond points, and thus has height $h := \mathsf{H}(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]}) \geq \delta n/10$. Rotating this walk clockwise by $\pi/3$ results in a walk of height at most h and width at least $h/2$ (due to the honeycomb geometry and the way we define height and width just above Lemma 11; the extreme case is reached for the bridge obtained by concatenating h copies of the bridge formed by a left turn followed by a right turn). Since the width of $\mathsf{StickBreak}_{i,j}(\gamma)$ must be greater than the width of any of its subwalks, in particular greater than the width of $\rho(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]})$, we must have

$$\mathsf{W}(\mathsf{StickBreak}_{i,j}(\gamma)) \geq \frac{h}{2} \geq \frac{\delta n}{20} > \varepsilon n,$$

since we have assumed $\varepsilon < \delta/20$. Since $\mathsf{H}(\rho(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]})) \leq \mathsf{H}(\gamma_{[\mathbf{d}_i,\mathbf{d}_j]})$, the $\mathsf{StickBreak}$ operation increases the height of γ by at most 1 (due to the attachment of the steps s and $\tilde{s}$). By (C_1), we then have $\mathsf{H}(\mathsf{StickBreak}_{i,j}(\gamma)) \leq \nu n + 1$ and therefore $\mathsf{StickBreak}_{i,j}(\gamma)$ has at most $\nu n + 2$ renewal points (as any subwalk between two renewal points must have height at least 1). Hence, for any $\phi \in \Omega^+$ such that $\mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi$, we have $\mathbf{r}_{\nu n+1} \geq |\mathsf{StickBreak}_{i,j}(\gamma)|$ and therefore

$$\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]}) \geq \mathsf{W}(\mathsf{StickBreak}_{i,j}(\gamma)) > \varepsilon n.$$

Thus, for any $(i,j,\gamma) \in \Phi$,

$$\begin{aligned} x_{\text{c}}^{|\mathsf{StickBreak}_{i,j}(\gamma)|} &= \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\phi \in \Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi\big) \\ &= \mathbb{P}_{\text{iSAB}}^{\otimes\mathbb{N}}\big(\phi \in \Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma) \lhd \phi \text{ and } \mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]}) > \varepsilon n\big). \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \sum_{(i,j,\gamma)\in\Phi} \mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\phi\in\Omega^+ : \mathsf{StickBreak}_{i,j}(\gamma)\lhd\phi \text{ and } \mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]}) > \varepsilon n\big) \\ &= \mathbb{E}^{\otimes\mathbb{N}}_{\text{iSAB}}\Big(\big|\{(i,j,\gamma)\in\Phi : \mathsf{StickBreak}_{i,j}(\gamma)\lhd\phi\}\big|\cdot \mathbb{1}_{\{\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]})>\varepsilon n\}}\Big) \\ &\leq (\nu n+2)^2\,\mathbb{P}^{\otimes\mathbb{N}}_{\text{iSAB}}\big(\mathsf{W}(\phi_{[0,\mathbf{r}_{\nu n+1}]}) > \varepsilon n\big). \end{aligned} \tag{29}$$

The last inequality follows from the fact that, for any given $\phi\in\Omega^+$, the number of elements (i,j,γ) of Φ such that $\mathsf{StickBreak}_{i,j}(\gamma)\lhd\phi$ is at most $(\nu n+2)^2$. Indeed, the triple (i,j,γ) is completely determined if we specify in ϕ the renewal point that precedes the step denoted s in (27) and the one that follows the step $\tilde{s}$. As both points occur before $\mathbf{r}_{\nu n+1}$, as explained above, the bound (29) follows.

By combining (28) and (29) we obtain (26), which concludes the proof. ■

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